



## Logarithmic Imputation Estimators for Domain Mean Estimation under Item Nonresponse

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### Abstract

Nonresponse is a persistent problem in survey sampling and can substantially reduce the efficiency of domain-specific estimates, particularly when item-nonresponse occurs within domains. Often, conventional approaches such as re-weighting, weight adjustments, calibration, and imputation often fail when response probabilities depend on the survey variable itself. This study develops alternative ratio-type logarithmic imputation estimators for estimating domain means to address item nonresponse and outlier problems under single-stage simple random sampling (SRS). Three imputation schemes are formulated according to the availability and use of auxiliary information, yielding three new estimators. First-order bias and mean squared error (MSE) expressions are derived using Taylor-series approximation, and theoretical efficiency conditions are established relative to selected existing estimators. An Empirical investigation based on household expenditure and income data from the 2020 Integrated Household Finance and Consumption Survey is conducted using the six geopolitical zones of Nigeria as domains and two item-nonresponse scenarios of approximately 60% and 40%. Across the empirical cases, the proposed estimators generally produced lower MSE and higher percentage relative efficiency than the competing estimators, with scheme 2 providing the strongest performance among the three schemes. The findings indicate that the alternative ratio-type logarithmic imputation estimator can improve the efficiency of domain mean estimation when extreme values and item-nonresponse are present. The study extends existing logarithmic imputation methodology by providing domain-specific estimators under alternative auxiliary-information conditions within a single-stage SRS framework.

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### 1.0 Introduction

Sample surveys have long been recognized as an effective means of obtaining information about

populations of interest. They are usually conducted on selected elements or units from a large study population to obtain accurate and

reliable information. According to Sarndal et al. (1992), the use of sample surveys gained prominence in the 1930s as a time- and cost-effective approach, while still providing reliable information about population characteristics through statistical inference. Sample surveys are conducted to collect data on a wide range of issues, including employment/unemployment, education, household income and expenditure, household size, housing, crime, agriculture, nutrition, and health conditions. Labour force surveys, for instance, provide data on employment, hours worked, industry of employment, and duties performed, which are used to monitor labour market conditions. Similarly, business surveys generate data on savings, investment, production, sales, job vacancies, and employment. Hospital surveys provide information on the number of doctors, nurses, and other caregivers, patient satisfaction, and hospital resources such as bed capacity. Environmental surveys supply data on air pollution, erosion, soil composition, and animal abundance, while customer surveys measure satisfaction with products and services.

Survey sampling is a cornerstone of statistical inference, enabling researchers to obtain reliable information about large populations at reduced cost and time. However, nonresponse, whether partial or complete, remains a major obstacle, introducing bias, inflating variance, and reducing efficiency. The problem is particularly acute in domain estimation, where sub-populations (e.g., by age, gender, or income level) require accurate estimates despite small sample sizes.

Early contributions by Yates (1953) and Hartley (1959) laid the foundation for domain estimation under single-stage and multi-stage sampling. Subsequent work by Hansen and Hurwitz (1946), Kalton and Kasprzyk (1986) and Tripathi (1988) explored subsampling and imputation strategies to mitigate nonresponse bias. More recent studies by Deville and Sarndal (1992), Rao et al. (2012), Iseh and Enang (2021),

Ekong, et al. (2021), Iseh and Bassey (2022, 2024) etc. have emphasized calibration and auxiliary information to compensate for small sample size in some domains of interest as well as improve on the precision and efficiency in domain estimation. Yet, when response probabilities depend on the survey variable itself, traditional methods often fail to eliminate bias.

Domain estimation has evolved through ratio, regression, synthetic, and calibration-based approaches. Cochran (1977) demonstrated variance reduction when domain sizes are known, while Agrawal and Midha (2007) advanced double sampling techniques. More recent contributions from Torabi et al. (2009), Ashutosh et al., (2019), Iseh and Bassey (2021), and Iseh et al. (2025) have highlighted the importance of auxiliary information and calibration in improving efficiency under nonresponse.

Gira (2015) proposed new imputation estimators for population mean with outstanding performance though not without some systematic bias. In the progression, Logarithmic-type estimators, introduced by Bhushan and Gupta (2015), have shown prominence in improving precision by leveraging auxiliary information. Parallel to these developments, logarithmic-type estimators have gained traction. Aslam et al. (2020), Noor-Ul-Amin (2020) and Das and Singh (2023) introduced memory-type estimators using exponentially weighted moving averages. Bhushan and Kumar (2022) further explored logarithmic predictive estimators under ranked set sampling. In the development, Bhushan et al. (2024) extended the Logarithmic-type EWMA estimators to domain estimation.

Despite these advances, limited attention has been given to logarithmic imputation strategies specifically in addressing domain-level nonresponse and extreme values or outliers. For instance, in household expenditure and income data, the presence of outlier in some domains of

interest with small sample size can result in unusually mean and variance inflations, and misleading forecasts. Although logarithmic imputation estimators have been developed for population mean estimation under nonresponse, and recent studies have extended imputation techniques to domain mean estimation, there remains a need for logarithmic-type estimators specifically formulated for domain means under item nonresponse and extreme values within a single-stage simple random sampling framework. The present study addresses this gap by developing and evaluating logarithmic imputation estimators under three auxiliary-information schemes.

## 2.0 Methodology

### 2.1 Sample design

Consider a finite population  $U = (U_1, U_2, \dots, U_N)$  of size  $N$ . Suppose the finite population under study  $U$  of size  $N$  is further divided into  $D$  domains;  $U_1, U_2, U_3, \dots, U_D$  of sizes;  $N_1, N_2, N_3, \dots, N_D$  respectively, from which a sample of size  $n_d$  is drawn according to simple random sampling without replacement (SRSWOR) design. The domain membership of any sub-unit and population unit will be known before sampling. It is assumed that domains may be quite large or small. Let  $y_{di}$  and  $x_{di}$  denote the values of the study and auxiliary variables for the  $i^{th}$  unit ( $i = 1, 2, \dots, N$ ) in the  $d^{th}$  domain of the population. Let  $\bar{Y} = \frac{1}{N} \sum_{i=1}^N y_i$  and  $\bar{y} = \frac{1}{n} \sum_{i=1}^n y_i$  be the unknown population mean and sample mean of the study variable  $y$  and  $\bar{X} = \frac{1}{N} \sum_{i=1}^N x_i$ ,  $\bar{x} = \frac{1}{n} \sum_{i=1}^n x_i$  respectively denotes the known population mean and sample mean of the auxiliary variable  $x$  respectively. For a typical  $d^{th}$  domain  $U_d$  several characteristics may be define including the domain total, mean, variance and covariance respectively as:

$$Y_{U_d} = \sum_{U_d} y_{di}, \quad \bar{Y}_{U_d} = \frac{1}{N_d} \sum_{U_d} y_{di},$$

$$S_{U_d}^2(Y) = \frac{1}{N_d-1} \sum_{k \in U_d} (y_{di} - \bar{Y}_{U_d})^2$$

$$C_{U_d}(X, Y) = \frac{1}{N_d-1} \sum_{k \in U_d} (x_{di} - \bar{X}_{U_d})(y_{di} - \bar{Y}_{U_d})$$

It is assumed that nonresponse is detected or noticed on the study variable  $Y$  only, and

auxiliary variable  $X$  is assumed to be free from nonresponse. Also, the case where nonresponse is observed in both the study and auxiliary variables will be considered. In view of the detection of nonresponse in the variable of interest and on both study and auxiliary variables, the proposed estimators are formulated in this regard with a view to compensating for nonresponse bias and its attendant negative effect on the efficiency estimators.

### 2.2 Study Notation

The following notations will be useful in the course of this work:

$N$  = Population size under study

$n$  = Sample size from the entire population

$X$  = Auxiliary variable

$Y$  = Study variable

$N_{td}$  = Population size of the  $d^{th}$  domain

$n_{td}$  = Sample size of the  $d^{th}$  domain

$\bar{y}_{td}$  = Unbiased estimator of the population mean for the  $d^{th}$  domain of the study variable.

$\bar{x}_{td}$  = Unbiased estimator of the population mean for the  $d^{th}$  domain of the auxiliary variable.

$C_{xtd}$  = Coefficient of variation for the  $d^{th}$  domain of the auxiliary variable.

$C_{ytd}$  = Coefficient of variation for the  $d^{th}$  domain of the study variable.

$\bar{Y}_{td}$  = Population mean for the  $d^{th}$  domain of the study variable.

$\bar{X}_{td}$  = Population mean for the  $d^{th}$  domain of the study variable. Other notations used in this study will be defined at their first mentioned.

### 2.3 Some Existing Imputation Estimators

#### Singh and Horn (2000) Estimator

Singh and Horn (2000) proposed a ratio point imputation estimator of population mean  $\bar{Y}$  given as:

$$\bar{y}_{sh} = \alpha \bar{y}_r + (1 - \alpha) \bar{y}_r \frac{\bar{x}_n}{\bar{x}_r} \quad (1)$$

$$\bar{y}_r = \frac{1}{n} \sum_{i=1}^n y_{ir}, \quad \bar{x}_n = \frac{1}{n} \sum_{i=1}^n x_i, \quad \bar{x}_r = \frac{1}{n} \sum_{i=1}^n x_{ir}$$

where  $r$  is the number of units responding from the sample size ( $n$ ),  $\alpha$  is a suitable constant  $\alpha = 1 - \rho_{xy} C_x C_y$ . The MSE is given as:

$$MSE(\bar{y}_{sh})_{opt} = MSE(\bar{y}_{rat}) - \left(\frac{1}{r} - \frac{1}{n}\right) (C_x - \rho_{xy} C_y)^2 \bar{Y}^2 \quad (2)$$

where;

$$MSE(\bar{y}_{rat}) = Var(\bar{y}_r) + \left(\frac{1}{r} - \frac{1}{n}\right) (C_x^2 - 2\rho_{xy} C_x C_y) \bar{Y}^2 \text{ and}$$

$$Var(\bar{y}_r) = \left(\frac{1}{r} - \frac{1}{N}\right) \frac{S_y^2}{\bar{Y}^2}, \quad S_y^2 = \frac{1}{N-1} \sum_{r=1}^N (y_r - \bar{Y})^2$$

### Bhushan and Gupta (2015) Estimator

Bhushan and Gupta (2015) proposed a logarithmic estimator under simple random sampling as follows:

$$\hat{\bar{y}}_a = \bar{y} \left[ 1 + \log \frac{\bar{x}}{\bar{X}} \right]^\lambda \quad (3)$$

where  $\lambda$  is a properly chosen scalar.

$$\min MSE(\hat{\bar{y}}_a) = \theta \bar{Y}^2 C_y^2 (1 - \rho_{xy}^2) \quad (4)$$

where  $\theta = \frac{1}{n}$

### Pandey et al. (2021) Estimator

Pandey et al (2021) proposed some class of logarithmic imputation estimators of population mean  $\bar{Y}$  given as:

$$T_{P1} = \bar{y}_r + \alpha \log(\bar{x}_n \bar{X}_d) \quad (5)$$

$$T_{P2} = \bar{y}_r + \beta \log(\bar{x}_r \bar{X}_d) \quad (6)$$

$$T_{P3} = \bar{y}_r + \gamma \log(\bar{x}_n \bar{x}_r) \quad (7)$$

$$\alpha = \beta = -\rho S_y C_x, \quad \gamma = \rho S_y C_x$$

Biases are given below;

$$B(T_{P1}) = \theta_2^2 \rho S_y C_x$$

$$B(T_{P2}) = \theta_1^2 \rho S_y C_x$$

$$B(T_{P3}) = \theta_3^2 \rho S_y C_x$$

The Mean Squared Errors are given as:

$$MSE(T_{P1}) = S_{2y} (\theta_1 - \theta_2 \rho^2) \quad (8)$$

$$MSE(T_{P2}) = \theta_1 S_{2y} (1 - \theta_2 \rho^2) \quad (9)$$

$$MSE(T_{P3}) = S_{2y} (\theta_1 - \theta_3 \rho^2) \quad (10)$$

### Bhushan et al. (2023) Estimator

Bhushan et al (2023) proposed some logarithmic type imputation estimators of population mean  $\bar{Y}$  given as:

$$T_{B1} = \psi_1 \bar{y}_r \left[ 1 + \log \left( \frac{Z_n}{\mu_Z} \right) \right]^{\lambda_1} \quad (11)$$

$$T_{B2} = \psi_2 \bar{y}_r \left[ 1 + \log \left( \frac{Z_r}{\mu_Z} \right) \right]^{\lambda_2} \quad (12)$$

$$T_{B3} = \psi_3 \bar{y}_r \left[ 1 + \log \left( \frac{Z_r}{Z_n} \right) \right]^{\lambda_3} \quad (13)$$

$\psi_i$  and  $\lambda_i, i = 1, 2, 3$  are properly chosen scalars.

$$\lambda_i(opt) = -\rho \frac{C_y}{\tau_z C_z} \text{ and } \psi_i(opt) = \frac{Q_i}{P_i}$$

The Mean Squared Errors are given as:

$$MSE(T_{Bi}) = \mu_y^2 [1 + \psi_i^2 P_i - 2\psi_i Q_i] \quad (14)$$

$$P_1 = 1 + f_r \tau_y C_y^2 + 2\lambda_1 (\lambda_1 - 1) f_n \tau_z C_z^2 + 4\lambda_1 f_n \rho C_y C_z$$

$$Q_1 = 1 + \frac{[\lambda_1 (\lambda_1 - 1)]}{2} f_n \tau_z C_z^2 + \lambda_1 f_n \rho C_y C_z$$

$$P_2 = 1 + f_r \tau_y C_y^2 + 2\lambda_2 (\lambda_2 - 1) f_r \tau_z C_z^2 + 4\lambda_2 f_r \rho C_y C_z$$

$$Q_2 = 1 + \frac{[\lambda_2 (\lambda_2 - 1)]}{2} f_r \tau_z C_z^2 + \lambda_2 f_r \rho C_y C_z$$

$$P_3 = 1 + f_r \tau_y C_y^2 + 2\lambda_3 (\lambda_3 - 1) f_{rn} \tau_z C_z^2 + 4\lambda_3 f_{rn} \rho C_y C_z$$

$$Q_3 = 1 + \frac{[\lambda_3 (\lambda_3 - 1)]}{2} f_{rn} \tau_z C_z^2 + \lambda_3 f_{rn} \rho C_y C_z$$

$$\tau_y = \frac{\sigma_y^2 + \sigma_u^2}{\sigma_y^2}, \quad \tau_z = \frac{\sigma_z^2 + \sigma_v^2}{\sigma_z^2}, \quad f_r = \frac{1}{r} - \frac{1}{N}, \quad f_n = \frac{1}{n} - \frac{1}{N}, \quad f_{rn} = \frac{1}{r} - \frac{1}{n}$$

### Das and Singh (2023) Estimator

Das & Singh (2023) proposed some efficient exponential-logarithmic imputation estimators of population mean  $\bar{Y}$ . The logarithmic imputation estimators are given as:

$$T_{DS1} = \bar{y}_r \left[ \exp \left[ \alpha \log \left[ 1 + \frac{\frac{a}{\bar{X}\bar{h}} - \frac{a}{\bar{x}_n\bar{h}}}{\frac{a}{\bar{X}\bar{h}} + \frac{a}{\bar{x}_n\bar{h}}} \right] \right] \right] \quad (15)$$

$$T_{DS2} = \bar{y}_r \left[ \exp \left[ \alpha \log \left[ 1 + \frac{\frac{a}{\bar{X}\bar{h}} - \frac{a}{\bar{x}_r\bar{h}}}{\frac{a}{\bar{X}\bar{h}} + \frac{a}{\bar{x}_r\bar{h}}} \right] \right] \right] \quad (16)$$

$$T_{DS3} = \bar{y}_r \left[ \exp \left[ \alpha \log \left[ 1 + \frac{\frac{a}{\bar{x}_n\bar{h}} - \frac{a}{\bar{x}_r\bar{h}}}{\frac{a}{\bar{x}_n\bar{h}} + \frac{a}{\bar{x}_r\bar{h}}} \right] \right] \right] \quad (17)$$

$$MSE(T_{DS1}) = \bar{Y}^2 (\lambda_{r,N} - \lambda_{n,N} \rho_{XY}^2) C_Y^2 \quad (18)$$

$$MSE(T_{DS2}) = \bar{Y}^2 \lambda_{r,N} (1 - \rho_{XY}^2) C_Y^2 \quad (19)$$

$$MSE(T_{DS3}) = \bar{Y}^2 (\lambda_{r,N} - \lambda_{r,n} \rho_{XY}^2) C_Y^2 \quad (20)$$

where  $\lambda_{r,N} = \frac{1}{r} - \frac{1}{N}$ ,  $\lambda_{r,n} = \frac{1}{r} - \frac{1}{n}$ ,  $\lambda_{n,N} = \frac{1}{n} - \frac{1}{N}$ ,

$$\alpha_{opt} = \frac{2h \phi_{XY}}{a} \text{ and } \phi_{XY} = \rho_{XY} \frac{C_Y}{C_X}$$

### Bhushan et al. (2024) Estimator

Bhushan et al (2024) proposed a class of logarithmic imputation estimators of population mean  $\bar{Y}$  given as:

$$T_{1BN2024} = \bar{y}_r \left[ 1 + \theta_1 \log \left( \frac{\bar{x}_n}{\bar{x}_d} \right) \right] \quad (21)$$

$$T_{2BN2024} = \bar{y}_r \left[ 1 + \theta_2 \log \left( \frac{\bar{x}_r}{\bar{x}_d} \right) \right] \quad (22)$$

The Mean Squared Errors are given as:

$$\text{Min } MSE(T_{1BN2024}) = \bar{Y}_d^2 C_y^2 [f_r - f_n \rho_{xy}^2] \quad (23)$$

$$\text{Min } MSE(T_{2BN2024}) = \bar{Y}_d^2 f_r C_y^2 [1 - \rho_{xy}^2] \quad (24)$$

### 2.3 Proposed Imputation Estimators

This work adopted and extended a logarithmic estimator proposed by Bhushan and Gupta (2015) and Bhushan et al. (2023) under simple random sampling to logarithmic imputation estimators in domain estimation using different statistical designs and phases as a way of curtailing the negative effect of nonresponse and extreme values on the efficiency of an estimator.

#### Sample Design for Imputation under Simple Random Sampling (SRS)

Consider a specified population  $U = \{1, 2, \dots, N\}$  of size  $N$  from which a random sample  $s$  of size  $n$  is drawn without replacement. In order to estimate the mean ( $\bar{y}_{dt}$ ) of domain  $d$ , the information generated in the sample is used. Let  $r_d$  and  $r$  be the number of units responding from chosen  $n_d$  and  $n$  units and let  $R_d$  and  $R$  be the set of units responding in the domain  $d$  and total population respectively. Also,  $R'_d$  and  $R'$  represent the set of units non-responding in the domain  $d$  and total population respectively. For all units,  $i \in R$ , the quantity  $y_i$  is obtained and for the units  $i \in R'$ , the quantities or observations are missing. Hence, the need to impute data to finalize the formation of sample dataset. Suppose, the imputation is accomplished comprising the additional auxiliary information  $X_t$ , such that, the value of  $X_t$  for unit  $i$  ( $X_{ti}$ ), is available and positive for all  $i \in s$ .

Hence, dataset  $X_{ts} = \{X_{ti} : i \in s\}$  are available.

### General Form

The imputation rule is expressed as:

$$y_{dri} = \begin{cases} y_{dri} & \text{if } i \in R \\ \frac{1}{n-r} [n\bar{y}_{dri} - r\bar{y}_{dri}] & \text{if } i \in R' \end{cases}$$

The point estimator of population mean takes the form:

$$\bar{y}_{dr} = \frac{1}{n} \sum_{i \in sn} y_{dri} = \frac{1}{n} [\sum_{i \in R} y_{dri} + \sum_{i \in R'} y_{dri}] \quad (25)$$

Here  $\bar{y}_{dri}$  denote the imputed value of the study variable corresponding to the  $i^{th}$  non-responding units in the domain. When ignoring the nonresponding group, then

$$\bar{y}_{dr} = \frac{1}{r} \sum_{i=1}^r y_{dri} \quad (26)$$

Utilization of additional auxiliary information leads to categorization of the imputation into three schemes.

Scheme 1: when  $\bar{X}_d$  known and  $\bar{x}_{dn}$  is utilized: This implies that the population auxiliary mean is known and used

Scheme 2: when  $\bar{X}_d$  known and  $\bar{x}_{dr}$  is utilized: This implies that the population auxiliary mean is known and the response-group auxiliary mean used.

Scheme 3: when  $\bar{X}_d$  unknown and  $\bar{x}_{dn}$  and  $\bar{x}_{dr}$  are utilized: This implies that the population auxiliary mean is unknown and the sample auxiliary information used.

Motivated by the contributions of Bhushan and Gupta (2015) and Bhushan et al. (2023), these schemes are proposed under the framework of simple random sampling. In application, the estimation of domain mean of expenditure pattern can be influenced by outliers as well as nonresponse. As such, the main purpose of this study is to construct stable and efficient estimators using ratio of the population and sample means of the auxiliary variable as coefficient for  $\bar{y}$  to minimize the influence of extreme values or outliers as in Bhushan et al. (2024) estimator.

Scheme 1:

$$\bar{y}_{dr1} = \begin{cases} y_{dri1} & \text{if } i \in R \\ \frac{1}{n-r} [n\Omega_0 [1 + \theta_1 \text{Log}[\Omega_1]]^\alpha - r\Omega_0 [1 + \theta_1 \text{Log}[\Omega_1]]^\alpha] & \text{if } i \in R' \end{cases}$$

Scheme 2:

$$\bar{y}_{dr2} = \begin{cases} y_{idr2} & \text{if } i \in R \\ \frac{1}{n-r} [n\Omega_0 [1 + \theta_2 \text{Log}[\Omega_2]]^\alpha - r\Omega_0 [1 + \theta_2 \text{Log}[\Omega_2]]^\alpha] & \text{if } i \in R' \end{cases}$$

Scheme 3:

$$\bar{y}_{dr3} = \begin{cases} y_{idr3} & \text{if } i \in R \\ \frac{1}{n-r} [n\Omega_0 [1 + \theta_3 \text{Log}[\Omega_3]]^\alpha - r\Omega_0 [1 + \theta_3 \text{Log}[\Omega_3]]^\alpha] & \text{if } i \in R' \end{cases}$$

where  $\Omega_0 = \frac{\bar{y}_{dr}}{\bar{x}_{dr}}$ ,  $\Omega_1 = \frac{\bar{x}_{dn}}{\bar{x}_d}$ ,  $\Omega_2 = \frac{\bar{x}_{dr}}{\bar{x}_d}$ ,  $\Omega_3 = \frac{\bar{x}_{dr}}{\bar{x}_{dn}}$

Also,  $\bar{y}_{dr1}$ ,  $\bar{y}_{dr2}$  and  $\bar{y}_{dr3}$  are the imputation estimators for scheme 1, scheme 2 and scheme 3 respectively. The proposed estimators with respect to the responding and nonresponding groups under the above schemes are:

$$\bar{y}_{dr1} = \tau_{r1}^{srs} = \Omega_0 [1 + \theta_1 \text{Log}[\Omega_1]]^\alpha \quad (27)$$

$$\bar{y}_{dr2} = \tau_{r2}^{srs} = \Omega_0 [1 + \theta_2 \text{Log}[\Omega_2]]^\alpha \quad (28)$$

$$\bar{y}_{dr3} = \tau_{r3}^{srs} = \Omega_0 [1 + \theta_3 \text{Log}[\Omega_3]]^\alpha \quad (29)$$

These are the logarithmic imputation estimators

### Properties of the Proposed Estimator

To obtain the bias and Mean Square Errors, the Taylor's series approximation method is considered. Consequently, the large sample approximation for the variable as follows:

$$\text{Let } e_0 = \frac{\bar{y}_{tdr} - \bar{Y}_{td}}{\bar{Y}_{td}}, e_1 = \frac{\bar{x}_{tdr} - \bar{X}_{td}}{\bar{X}_{td}}, e_2 = \frac{\bar{x}_{tdn} - \bar{X}_{td}}{\bar{X}_{td}}$$

$$\text{where } \bar{y}_{tdr} = \frac{\sum_{i=1}^{r_{td}} \bar{y}_{tdri}}{r_{td}}, \bar{x}_{tdr} = \frac{\sum_{i=1}^{r_{td}} \bar{x}_{tdri}}{r_{td}}, \bar{x}_{tdn} = \frac{\sum_{i=1}^{n_{td}} \bar{x}_{tdni}}{n_{td}}, \bar{X}_{td} = \frac{\sum_{i=1}^{N_{td}} \bar{X}_{tdi}}{N_{td}}, \bar{Y}_{td} = \frac{\sum_{i=1}^{N_{td}} \bar{Y}_{tdi}}{N_{td}}$$

Also,

$$\bar{y}_{tdr} = \bar{Y}_{td}(1 + e_0), \quad \bar{x}_{tdr} = \bar{X}_{td}(1 + e_1), \quad \bar{x}_{tdn} = \bar{X}_{td}(1 + e_2) \quad (30)$$

$$E[e_0] = E\left[\frac{\bar{y}_{tdr} - \bar{Y}_{td}}{\bar{Y}_{td}}\right] = \frac{\bar{Y}_{td} - \bar{Y}_{td}}{\bar{Y}_{td}} = 0$$

$$\text{Similarly, } E[e_1] = E[e_2] = 0$$

$$\therefore E[e_0] = E[e_1] = E[e_2] = 0$$

In summary, the expectation of the large sample approximation for the variables are given in Table 1.

**Table 1:** Expectations for the Large Sample Approximations

| EXPECTATIONS |       |       |       |                 |                 |                 |                            |                            |                 |
|--------------|-------|-------|-------|-----------------|-----------------|-----------------|----------------------------|----------------------------|-----------------|
| SCHEMES      | $e_0$ | $e_1$ | $e_2$ | $e_0^2$         | $e_1^2$         | $e_2^2$         | $e_0 e_1$                  | $e_0 e_2$                  | $e_1 e_2$       |
| 1            | 0     | 0     | 0     | $F_1 C_{ytd}^2$ | $F_1 C_{xtd}^2$ | $F_2 C_{xtd}^2$ | $F_1 \rho C_{xtd} C_{ytd}$ | $F_2 \rho C_{xtd} C_{ytd}$ | $F_2 C_{xtd}^2$ |
| 2            | 0     | 0     | 0     | $F_1 C_{ytd}^2$ | $F_1 C_{xtd}^2$ | $F_2 C_{xtd}^2$ | $F_1 \rho C_{xtd} C_{ytd}$ | $F_2 \rho C_{xtd} C_{ytd}$ | $F_2 C_{xtd}^2$ |
| 3            | 0     | 0     | 0     | $F_1 C_{ytd}^2$ | $F_1 C_{xtd}^2$ | $F_2 C_{xtd}^2$ | $F_1 \rho C_{xtd} C_{ytd}$ | $F_2 \rho C_{xtd} C_{ytd}$ | $F_2 C_{xtd}^2$ |

Note:  $F_1 = \left(\frac{1}{r_{td}} - \frac{1}{n_{td}}\right)$ ,  $F_2 = \left(\frac{1}{n_{td}} - \frac{1}{N_{td}}\right)$ ,  $F_3 = F_1 - F_2 = \left(\frac{1}{r_{td}} - \frac{1}{n_{td}}\right)$

Let  $A_1 = [C_{xtd}^2 - 2\rho C_{xtd}C_{ytd}]$ ,  $A_2 = [C_{xtd}^2 - \rho C_{xtd}C_{ytd}]$  and  $A_3 = [C_{xtd}^2 + \rho C_{xtd}C_{ytd}]$

### Properties of the proposed logarithmic imputation estimator for Scheme 1

**Theorem 1:** The proposed logarithmic imputation estimator  $\tau_{r1}^{srS}$  for Scheme 1 is biased and has optimum MSE respectively as follows:  $Bias[\tau_{r1}^{srS}] = \bar{Y}_{td} \left[ A_2(F_1 - \alpha\theta_1 F_2) - \frac{\theta_1 \alpha^2}{2} F_2 C_{xtd}^2 \right]$

and  $MSE_{opt}(\tau_{r1}^{srS}) = \bar{Y}_{td}^2 \left[ F_1(C_{ytd}^2 + A_1) - \frac{F_2[A_2]^2}{C_{xtd}^2} \right]$

Proof:

Recall (27)

$$\bar{y}_{dr1} = \tau_{r1}^{srS} = \frac{\bar{y}_{tdr}}{\bar{x}_{tdr}} \bar{X}_{td} \left[ 1 + \theta_1 \text{Log} \left[ \frac{\bar{x}_{tdn}}{\bar{X}_{td}} \right] \right]^\alpha$$

Substituting (30) in (27)

$$\begin{aligned} \tau_{r1}^{srS} &= \frac{\bar{Y}_{td}(1+e_0)}{\bar{X}_{td}(1+e_1)} \bar{X}_{td} \left[ 1 + \theta_1 \text{Log} \left[ \frac{\bar{X}_{td}(1+e_2)}{\bar{X}_{td}} \right] \right]^\alpha \\ &= \frac{\bar{Y}_{td}(1+e_0)}{(1+e_1)} [1 + \theta_1 \text{Log}[(1+e_2)]]^\alpha \\ &= \bar{Y}_{td}(1+e_0)(1+e_1)^{-1} [1 + \theta_1 \text{Log}[(1+e_2)]]^\alpha \\ &= \bar{Y}_{td}(1+e_0)(1-e_1+e_1^2-\dots) [1 + \theta_1 \text{Log}[(1+e_2)]]^\alpha \end{aligned}$$

But  $\log(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} \dots$

$$= \bar{Y}_{td}(1+e_0)(1-e_1+e_1^2-\dots) \left[ 1 + \theta_1 \left[ e_2 - \frac{e_2^2}{2} + \frac{e_2^3}{3} \right] \right]^\alpha$$

But  $Bias(\tau_{r1}^{srS}) = E[\tau_{r1}^{srS} - \bar{Y}_{td}]$ , thus;

$$\begin{aligned} E[\tau_{r1}^{srS} - \bar{Y}_{td}] &= E \left[ \bar{Y}_{td} \left[ 1 + \alpha\theta_1 e_2 - \frac{\theta_1 \alpha^2 e_2^2}{2} - e_1 - \alpha\theta_1 e_1 e_2 + e_1^2 + e_0 + \alpha\theta_1 e_0 e_2 - e_0 e_1 - 1 \right] \right] \\ &= E \left[ \bar{Y}_{td} \left[ \alpha\theta_1 e_2 - \frac{\theta_1 \alpha^2 e_2^2}{2} - e_1 - \alpha\theta_1 e_1 e_2 + e_1^2 + 2e_0 + \alpha\theta_1 e_0 e_2 - e_0 e_1 \right] \right] \\ Bias(\tau_{r1}^{srS}) &= \bar{Y}_{td} \left[ A_2(F_1 - \alpha\theta_1 F_2) - \frac{\theta_1 \alpha^2}{2} F_2 C_{xtd}^2 \right] \end{aligned} \quad (31)$$

Mean Square Error is given as  $E[\tau_{r1}^{srS} - \bar{Y}_{td}]^2$ , thus;

$$\begin{aligned} E[\tau_{r1}^{srS} - \bar{Y}_{td}]^2 &= E[\bar{Y}_{td}[\alpha\theta_1 e_2 - e_1 + e_0]]^2 \\ &= \bar{Y}_{td}^2 \left[ \alpha^2 \theta_1^2 E[e_2^2] - 2\alpha\theta_1 [E[(e_1 e_2)] - E[e_0 e_2]] - 2E[e_0 e_1] + E[e_0^2] + E[e_1^2] \right] \\ MSE(\tau_{r1}^{srS}) &= \bar{Y}_{td}^2 [\alpha^2 \theta_1^2 F_2 C_{xtd}^2 - 2\alpha\theta_1 F_2 A_2 + F_1(C_{ytd}^2 + A_1)] \end{aligned} \quad (32)$$

### Determination of optimum mean squared error of $\tau_{r1}^{srS}$

To obtain the optimum value for the mean squared error of  $\tau_{r1}^{srS}$ , Eq. 32 is differentiated partially with respect to  $\alpha$

$$\frac{\partial(MSE(\tau_{r1}^{srs}))}{\partial \alpha} = 2\alpha\theta_1^2 F_2 C_{xtd}^2 - 2F_2\theta_1 A_2 = 0$$

$$\hat{\alpha} = \frac{A_2}{\theta_1 C_{xtd}^2}$$

Substituting the value  $\hat{\alpha} = \frac{A_2}{\theta_1 C_{xtd}^2}$  in Eq. 32 gives the optimum of  $MSE(\tau_{r1}^{srs})$  given as:

$$SE(\tau_{r1}^{srs})_{opt} = \bar{Y}_{td}^2 \left[ F_1 (C_{ytd}^2 + A_1) - \frac{F_2 [A_2]^2}{C_{xtd}^2} \right] \quad (33)$$

“The bias and the optimum MSE are given in equations (31) and (33), respectively; hence, Theorem 1 holds.”

### Properties of the proposed logarithmic imputation for Scheme 2

**Theorem 2:** The proposed logarithmic imputation estimator  $\tau_{r2}^{srs}$  under SRS for Scheme 2 is biased and has optimum MSE as follows:  $Bias(\tau_{r1}^{srs}) = \bar{Y}_{td} \left[ A_1(F_1 - \alpha\theta_2 F_1) - \frac{\theta_2 \alpha^2}{2} F_1 C_{xtd}^2 \right]$  and

$$MSE(\tau_{r2}^{srs})_{opt} = \bar{Y}_{td}^2 \left[ F_1 (C_{ytd}^2 + A_1) - \frac{F_1 [A_2]^2}{C_{xtd}^2} \right].$$

Proof:

Recall (28)

$$\tau_{r2}^{srs} = \frac{\bar{y}_{tdr}}{\bar{x}_{tdr}} \bar{X}_{td} \left[ 1 + \theta_2 \text{Log} \left[ \frac{\bar{x}_{tdr}}{\bar{X}_{td}} \right] \right]^\alpha$$

Substituting (30) in (28)

$$\begin{aligned} \tau_{r2}^{srs} &= \frac{\bar{Y}_{td}(1+e_0)}{\bar{X}_{td}(1+e_1)} \bar{X}_{td} \left[ 1 + \theta_2 \text{Log} \left[ \frac{\bar{X}_{td}(1+e_1)}{\bar{X}_{td}} \right] \right]^\alpha \\ &= \frac{\bar{Y}_{td}(1+e_0)}{(1+e_1)} [1 + \theta_2 \text{Log}[(1+e_1)]]^\alpha \\ &= \bar{Y}_{td}(1+e_0)(1+e_1)^{-1} [1 + \theta_2 \text{Log}[(1+e_1)]]^\alpha \\ &= \bar{Y}_{td}(1+e_0)(1-e_1+e_1^2-\dots) [1 + \theta_2 \text{Log}[(1+e_1)]]^\alpha \\ &= \bar{Y}_{td}(1+e_0)(1-e_1+e_1^2-\dots) \left[ 1 + \theta_2 \left[ e_1 - \frac{e_1^2}{2} + \frac{e_1^3}{3} \right] \right]^\alpha \end{aligned}$$

And the Bias is given as;

$$\begin{aligned} Bias(\tau_{r2}^{srs}) &= E \left[ \bar{Y}_{td} \left[ 1 + \alpha\theta_2 e_1 - \frac{\theta_2 \alpha^2 e_1^2}{2} - e_1 - \alpha\theta_2 e_1^2 + e_1^2 + e_0 + \alpha\theta_2 e_0 e_1 - e_0 e_1 - 1 \right] \right] \\ &= E \left[ \bar{Y}_{td} \left[ \alpha\theta_2 e_1 - \frac{\theta_2 \alpha^2 e_1^2}{2} - e_1 - \alpha\theta_2 e_1^2 + e_1^2 + 2e_0 + \alpha\theta_2 e_0 e_1 - e_0 e_1 \right] \right] \\ &= \bar{Y}_{td} \left[ -\frac{\theta_2 \alpha^2}{2} F_1 C_{xtd}^2 - \alpha\theta_2 F_1 C_{xtd}^2 + F_1 C_{xtd}^2 + \alpha\theta_2 F_1 \rho C_{xtd} C_{ytd} - F_1 \rho C_{xtd} C_{ytd} \right] \\ &= \bar{Y}_{td} \left[ A_1(F_1 - \alpha\theta_2 F_1) - \frac{\theta_2 \alpha^2}{2} F_1 C_{xtd}^2 \right] \quad (34) \end{aligned}$$

$$\begin{aligned} E[\tau_{r2}^{srs} - \bar{Y}_{td}]^2 &= E[\bar{Y}_{td} [\alpha\theta_2 e_1 - e_1 + e_0]]^2 \\ &= \bar{Y}_{td}^2 \left[ \alpha^2 \theta_2^2 E[e_1^2] - 2\alpha\theta_2 [E[e_1^2] - E[e_0 e_1]] - 2E[e_0 e_1] + E[e_0^2] + E[e_1^2] \right] \\ &= \bar{Y}_{td}^2 \left[ \alpha^2 \theta_2^2 F_1 C_{xtd}^2 - 2\alpha\theta_2 [F_1 C_{xtd}^2 - F_1 \rho C_{xtd} C_{ytd}] - 2F_1 \rho C_{xtd} C_{ytd} + F_1 C_{ytd}^2 + F_1 C_{xtd}^2 \right] \\ MSE(\tau_{r2}^{srs}) &= \bar{Y}_{td}^2 \left[ F_1 (C_{ytd}^2 + A_1) + \alpha^2 \theta_2^2 F_1 C_{xtd}^2 - 2\alpha\theta_2 F_1 [A_2] \right] \quad (35) \end{aligned}$$

### Determination of optimum mean squared error of $\tau_{r2}^{srs}$

To obtain the optimum value for the mean squared error of  $\tau_{r2}^{srs}$ ,  $MSE(\tau_{r2}^{srs})$  is differentiated partially with respect to  $\alpha$

$$\frac{\partial(MSE(\tau_{r2}^{srs}))}{\partial\alpha} = 2\alpha\theta_2^2 F_1 C_{xtd}^2 - 2F_1\theta_2 [A_2] = 0$$

$$\hat{\alpha} = \frac{[A_2]}{\theta_2 C_{xtd}^2}$$

Substituting the value  $\hat{\alpha} = \frac{[A_2]}{\theta_2 C_{xtd}^2}$  in Eq. 35 gives the optimum of  $MSE(\tau_{r2}^{srs})$  given as

$$MSE(\tau_{r2}^{srs})_{opt} = \left[ F_1 (C_{ytd}^2 + A_1) - \frac{F_1 [A_2]^2}{C_{xtd}^2} \right] \quad (36)$$

The bias and the optimum MSE are given in equations (34) and (36), respectively; hence, Theorem 2 holds.

### Properties of the proposed logarithmic imputation estimator under SRS for Scheme 3

**Theorem 3:** The proposed logarithmic imputation estimator  $\tau_{r3}^{srs}$  under SRS for Scheme 3 is biased and has optimum MSE as follows:  $Bias(\tau_{r3}^{srs}) = \bar{Y}_{td} \left[ (F_1 - F_3\alpha\theta_3) A_2 - \frac{\theta_3\alpha^2}{2} F_3 C_{xtd}^2 \right]$  and

$$MSE(\tau_{r3}^{srs})_{opt} = \bar{Y}_{td}^2 \left[ F_1 (C_{ytd}^2 + A_1) - \frac{F_3 [A_3]^2}{C_{xtd}^2} \right].$$

Proof:

Recall (29)

$$\bar{y}_{dr3} = \tau_{r3}^{srs} = \Omega_0 [1 + \theta_3 \text{Log}[\Omega_3]]^\alpha$$

Substituting (30) in (29)

$$\begin{aligned} \tau_{r3}^{srs} &= \frac{\bar{Y}_{td}(1+e_0)}{\bar{X}_{td}(1+e_1)} \bar{X}_{td} \left[ 1 + \theta_3 \text{Log} \left[ \frac{\bar{X}_{td}(1+e_1)}{\bar{X}_{td}(1+e_2)} \right] \right]^\alpha = \frac{\bar{Y}_{td}(1+e_0)}{(1+e_1)} \left[ 1 + \theta_3 \text{Log} \left[ \frac{(1+e_1)}{(1+e_2)} \right] \right]^\alpha \\ &= \bar{Y}_{td}(1+e_0)(1+e_1)^{-1} [1 + \theta_3 [\text{Log}(1+e_1) - \text{Log}(1+e_2)]]^\alpha \\ &= \bar{Y}_{td}(1+e_0)(1-e_1+e_1^2-\dots) \left[ 1 + \theta_3 \left[ e_1 - \frac{e_1^2}{2} + \frac{e_1^3}{3} \right] - \theta_3 \left[ e_2 - \frac{e_2^2}{2} + \frac{e_2^3}{3} \right] \right]^\alpha \\ &= \bar{Y}_{td} [1 - e_1 + e_1^2 + e_0 - e_0e_1 + e_1^2e_0] \left[ 1 - \alpha\theta_3e_2 + \frac{\theta_3\alpha^2e_2^2}{2} + \alpha\theta_3e_1 - \frac{\theta_3\alpha^2e_1^2}{2} \right] \end{aligned}$$

Bias is given as  $E[\tau_{r3}^{srs} - \bar{Y}_{td}]$ , thus;

$$\begin{aligned} E[\tau_{r3}^{srs} - \bar{Y}_{td}] &= E \left[ \bar{Y}_{td} \left[ 1 + e_0 - \alpha\theta_3e_2 + \frac{\theta_3\alpha^2e_2^2}{2} + \alpha\theta_3e_1 - \frac{\theta_3\alpha^2e_1^2}{2} - e_1 + \alpha\theta_3e_1e_2 - \alpha\theta_3e_1^2 \right. \right. \\ &\quad \left. \left. + e_1^2 - \alpha\theta_3e_0e_2 + \alpha\theta_3e_0e_1 - e_0e_1 - 1 \right] \right] \\ &= \bar{Y}_{td} \left[ \frac{\theta_3\alpha^2}{2} E[e_2^2] - \frac{\theta_3\alpha^2}{2} E[e_1^2] + \alpha\theta_3 E[e_1e_2] - \alpha\theta_3 E[e_1^2] + E[e_1^2] - \alpha\theta_3 E[e_0e_2] + \right. \\ &\quad \left. \alpha\theta_3 E[e_0e_1] - E[e_0e_1] \right] \end{aligned}$$

$$\text{Bias}(\tau_{r3}^{srS}) = \bar{Y}_{td} \left[ (F_1 - F_3 \alpha \theta_3) A_2 - \frac{\theta_3 \alpha^2}{2} F_3 C_{xtd}^2 \right] \quad (37)$$

Mean Square Error is given as  $E[\tau_{r3}^{srS} - \bar{Y}_{td}]^2$ , thus;

$$\begin{aligned} E[\tau_{r3}^{srS} - \bar{Y}_{td}]^2 &= E[\bar{Y}_{td}[-\alpha \theta_3 e_2 + \alpha \theta_3 e_1 - e_1 + e_0]]^2 \\ &= \bar{Y}_{td}^2 E[\alpha^2 \theta_3^2 e_2^2 - \alpha^2 \theta_3^2 e_1 e_2 + \alpha \theta_3 e_1 e_2 - \alpha \theta_3 e_0 e_2 - \alpha^2 \theta_3^2 e_1 e_2 + \alpha^2 \theta_3^2 e_1^2 - \alpha \theta_3 e_1^2 + \alpha \theta_3 e_0 e_1 \\ &\quad + \alpha \theta_3 e_1 e_2 - \alpha \theta_3 e_1^2 + e_1^2 - e_0 e_1 - \alpha \theta_3 e_0 e_2 + \alpha \theta_3 e_0 e_1 - e_0 e_1 + e_0^2] \\ &= \bar{Y}_{td}^2 [\alpha^2 \theta_3^2 E[e_2^2] + \alpha^2 \theta_3^2 E[e_1^2] - 2\alpha \theta_3 E[e_1^2] - 2\alpha^2 \theta_3^2 E[e_1 e_2] + 2\alpha \theta_3 E[e_1 e_2] - 2\alpha \theta_3 E[e_0 e_2] \\ &\quad + 2\alpha \theta_3 E[e_0 e_1] - 2E[e_0 e_1] + E[e_0^2] + E[e_1^2]] \\ &= \bar{Y}_{td}^2 [F_1 (C_{ytd}^2 + A_1) + F_3 \alpha^2 \theta_3^2 C_{xtd}^2 - 2F_3 \alpha \theta_3 A_3] \\ \text{MSE}(\tau_{r3}^{srS}) &= \bar{Y}_{td}^2 [F_1 (C_{ytd}^2 + A_1) + F_3 \alpha^2 \theta_3^2 C_{xtd}^2 - 2F_3 \alpha \theta_3 A_3] \end{aligned} \quad (38)$$

### Determination of optimum mean squared error of $\tau_{r3}^{srS}$

To obtain the optimum value for the mean squared error of  $\tau_{r3}^{srS}$ , Eq.38 is differentiated partially with respect to  $\alpha$

$$\frac{\partial(\text{MSE}(\tau_{r3}^{srS}))}{\partial \alpha} = 2\alpha \theta_3^2 C_{xtd}^2 F_3 - 2F_3 \theta_3 A_3 = 0$$

$$\hat{\alpha} = \frac{A_3}{\theta_3 C_{xtd}^2}$$

Substituting the value  $\alpha$  of in Eq. 38 gives the optimum  $\text{MSE}(\tau_{r3}^{srS})$  as follows

$$\text{MSE}(\tau_{r3}^{srS})_{opt} = \bar{Y}_{td}^2 \left[ F_1 (C_{ytd}^2 + A_1) - \frac{F_3 [A_3]^2}{C_{xtd}^2} \right] \quad (39)$$

“The bias and the optimum MSE are given in equations (37) and (39), respectively; hence, Theorem 3 holds.”

### 3.0 Efficiency Comparison of the Estimators

The efficiency of the existing logarithmic imputation estimators [Pandey et al. (2021), Das and Singh (2023), Bhushan et al. (2023)] under different schemes is compared with the proposed logarithmic imputation estimators. Since the existing estimators are based on simple random sampling designs, the comparisons focused on the estimators under the corresponding sampling framework in single-stage sampling design with respect to the schemes. The comparisons are expressed in the form of Propositions.

**Proposition 1:**  $\text{MSE}(\tau_{r1}^{srS})_{opt} - \text{MSE}(T_{B1}) < 0$  if

$$\begin{aligned} \bar{Y}_{td}^2 \left[ F_1 (C_{ytd}^2 + A_1) - \frac{F_2 [A_2]^2}{C_{xtd}^2} \right] - \mu_y^2 [1 + \psi_i^2 P_i - 2\psi_i Q_i] &< 0 \text{ when } i = 1 \\ \left[ F_1 (C_{ytd}^2 + A_1) - \frac{F_2 [A_2]^2}{C_{xtd}^2} \right] - [1 + \psi_i^2 P_i - 2\psi_i Q_i] &< 0 \end{aligned} \quad (40)$$

If (40) is satisfied, then the proposed estimator is more efficient than the existing estimator.

**Proposition 2:**  $\text{MSE}(\tau_{r2}^{srS})_{opt} - \text{MSE}(T_{B2}) < 0$  if

$$\begin{aligned} \bar{Y}_{td}^2 \left[ F_1 (C_{ytd}^2 + A_1) - \frac{F_1 [A_2]^2}{C_{xtd}^2} \right] - \mu_y^2 [1 + \psi_i^2 P_i - 2\psi_i Q_i] &< 0 \text{ when } i = 2 \\ \left[ F_1 (C_{ytd}^2 + A_1) - \frac{F_1 [A_2]^2}{C_{xtd}^2} \right] - [1 + \psi_i^2 P_i - 2\psi_i Q_i] &< 0 \end{aligned} \quad (41)$$

If (41) is satisfied, then the proposed estimator is more efficient than the existing estimator.

**Proposition 3:**  $MSE(\tau_{r3}^{srS})_{opt} - MSE(T_{B3}) < 0$  if

$$\begin{aligned} & \bar{Y}_{td}^2 \left[ F_1(C_{ytd}^2 + A_1) - \frac{F_3[A_3]^2}{C_{xtd}^2} \right] - \mu_y^2 [1 + \psi_i^2 P_i - 2\psi_i Q_i] < 0 \text{ when } i = 3 \\ & \left[ F_1(C_{ytd}^2 + A_1) - \frac{F_3[A_3]^2}{C_{xtd}^2} \right] - [1 + \psi_i^2 P_i - 2\psi_i Q_i] < 0 \end{aligned} \quad (42)$$

If (42) is satisfied, then the proposed estimator is more efficient than the existing estimator.

**Proposition 4:**  $MSE(\tau_{r1}^{srS})_{opt} - MSE(T_{P1}) < 0$  if

$$\begin{aligned} & \bar{Y}_{td}^2 \left[ F_1(C_{ytd}^2 + A_1) - \frac{F_2[A_2]^2}{C_{xtd}^2} \right] - S_{2y}(\theta_1 - \theta_2 \rho^2) < 0 \\ & -S_{2y}(\theta_1 - \theta_2 \rho^2) < -\bar{Y}_{td}^2 \left[ F_1(C_{ytd}^2 + A_1) - \frac{F_2[A_2]^2}{C_{xtd}^2} \right] \\ & \theta_2 \rho^2 < -\bar{Y}_{td}^2 \left[ F_1(C_{ytd}^2 + A_1) - \frac{F_2[A_2]^2}{C_{xtd}^2} \right] + S_{2y} \theta_1 \\ & \rho^2 < \frac{-\bar{Y}_{td}^2 \left[ F_1(C_{ytd}^2 + A_1) - \frac{F_2[A_2]^2}{C_{xtd}^2} \right] + S_{2y} \theta_1}{\theta_2} \end{aligned} \quad (43)$$

If (43) is satisfied, then the proposed estimator is more efficient than the existing estimator.

**Proposition 5:**  $MSE(\tau_{r2}^{srS})_{opt} - MSE(T_{P2}) < 0$  if

$$\begin{aligned} & \bar{Y}_{td}^2 \left[ F_1(C_{ytd}^2 + A_1) - \frac{F_1[A_2]^2}{C_{xtd}^2} \right] - \theta_1 S_{2y} (1 - \theta_2 \rho^2) < 0 \\ & \theta_1 \theta_2 \rho^2 < -\bar{Y}_{td}^2 \left[ F_1(C_{ytd}^2 + A_1) - \frac{F_1[A_2]^2}{C_{xtd}^2} \right] + \theta_1 S_{2y} \\ & \rho^2 < \frac{-\bar{Y}_{td}^2 \left[ F_1(C_{ytd}^2 + A_1) - \frac{F_1[A_2]^2}{C_{xtd}^2} \right] + \theta_1 S_{2y}}{\theta_1 \theta_2} \end{aligned} \quad (44)$$

If (44) is satisfied, then the proposed estimator is more efficient than the existing estimator.

**Proposition 6:**  $MSE(\tau_{r3}^{srS})_{opt} - MSE(T_{P3}) < 0$  if

$$\begin{aligned} & \bar{Y}_{td}^2 \left[ F_1(C_{ytd}^2 + A_1) - \frac{F_3[A_3]^2}{C_{xtd}^2} \right] - S_{2y}(\theta_1 - \theta_3 \rho^2) < 0 \\ & \theta_3 S_{2y} \rho^2 < -\bar{Y}_{td}^2 \left[ F_1(C_{ytd}^2 + A_1) - \frac{F_3[A_3]^2}{C_{xtd}^2} \right] + \theta_1 S_{2y} \\ & \rho^2 < \frac{-\bar{Y}_{td}^2 \left[ F_1(C_{ytd}^2 + A_1) - \frac{F_3[A_3]^2}{C_{xtd}^2} \right] + \theta_1 S_{2y}}{\theta_3 S_{2y}} \end{aligned} \quad (45)$$

If (45) is satisfied, then the proposed estimator is more efficient than the existing estimator.

**Proposition 7:**  $MSE(\tau_{r1}^{srS})_{opt} - MSE(T_{DS1}) < 0$  if

$$\begin{aligned} & \bar{Y}_{td}^2 \left[ F_1(C_{ytd}^2 + A_1) - \frac{F_2[A_2]^2}{C_{xtd}^2} \right] - \bar{Y}^2 (\lambda_{r,N} - \lambda_{n,N} \rho_{XY}^2) C_Y^2 < 0 \\ & \lambda_{n,N} \rho_{XY}^2 C_Y^2 < -\left[ F_1(C_{ytd}^2 + A_1) - \frac{F_2[A_2]^2}{C_{xtd}^2} \right] + \lambda_{r,N} C_Y^2 \\ & \rho^2 < \frac{-\left[ F_1(C_{ytd}^2 + A_1) - \frac{F_2[A_2]^2}{C_{xtd}^2} \right] + \lambda_{r,N} C_Y^2}{\lambda_{n,N} C_Y^2} \end{aligned} \quad (46)$$

If (46) is satisfied, then the proposed estimator is more efficient than the existing estimator.

**Proposition 8:**  $MSE(\tau_{r2}^{srS})_{opt} - MSE(T_{DS2}) < 0$  if

$$\begin{aligned}
& \bar{Y}_{td}^2 \left[ F_1(C_{ytd}^2 + A_1) - \frac{F_1[A_2]^2}{C_{xtd}^2} \right] - \bar{Y}^2 \lambda_{r,N} (1 - \rho_{XY}^2) C_Y^2 < 0 \\
& \rho_{XY}^2 C_Y^2 < - \left[ F_1(C_{ytd}^2 + A_1) - \frac{F_1[A_2]^2}{C_{xtd}^2} \right] + \lambda_{r,N} C_Y^2 \\
& \rho_{XY}^2 < \frac{- \left[ F_1(C_{ytd}^2 + A_1) - \frac{F_1[A_2]^2}{C_{xtd}^2} \right] + \lambda_{r,N} C_Y^2}{C_Y^2}
\end{aligned} \tag{47}$$

If (47) is satisfied, then the proposed estimator is more efficient than the existing estimator.

**Proposition 9:**  $MSE(\tau_{r3}^{srS})_{opt} - MSE(T_{DS3}) < 0$  if

$$\begin{aligned}
& \bar{Y}_{td}^2 \left[ F_1(C_{ytd}^2 + A_1) - \frac{F_3[A_3]^2}{C_{xtd}^2} \right] - \bar{Y}^2 (\lambda_{r,N} - \lambda_{r,n} \rho_{XY}^2) C_Y^2 < 0 \\
& \lambda_{r,n} C_Y^2 \rho_{XY}^2 < - \left[ F_1(C_{ytd}^2 + A_1) - \frac{F_3[A_3]^2}{C_{xtd}^2} \right] + \lambda_{r,N} C_Y^2 \\
& \rho^2 < \frac{- \left[ F_1(C_{ytd}^2 + A_1) - \frac{F_3[A_3]^2}{C_{xtd}^2} \right] + \lambda_{r,N} C_Y^2}{\lambda_{r,n} C_Y^2}
\end{aligned} \tag{48}$$

If (48) is satisfied, then the proposed estimator is more efficient than the existing estimator.

#### 4.0 Empirical Investigation

To illustrate the performance of the estimators, data from the Department of Statistics, Central Bank of Nigeria (CBN) were used. This real-life dataset was obtained from the Integrated Household Finance & Consumption Survey (IHFCS) conducted in 2020. The study focused on Household Finance and Consumption (HFC), which provides quarterly information on the assets and liabilities of households and members aged 18 years and above, commonly referred to as a wave. The IHFCS employs a multi-stage probability sampling procedure for households and their respective enumeration areas. The survey is conducted every quarter (four waves per year) as part of the routine data generation activities of the CBN. In this study, each quarter is regarded as a subpopulation, and the data were pooled into the six geopolitical zones of Nigeria, which serve as the domains: North Central (7 States) – D<sub>1</sub>, North East (6 States) – D<sub>2</sub>, North West (7 States) – D<sub>3</sub>, South East (5 States) – D<sub>4</sub>, South South (6 States) – D<sub>5</sub>, and South West (6 States) – D<sub>6</sub>. The quarters and domains are represented by the notation:  $Q_i D_j$  ( $i = 1, 2, 3$  and  $4, j = 1, 2, 3, 4, 5$  and  $6$ ). To illustrate an ideal imputation scenario, the study variable Y is defined as household expenditure, while the auxiliary variable X is household income. The objective is to estimate the mean household expenditure for the six domains using household income. Two item-response rates are considered, Cases I and II. Case I: If the rate of item response available in the domains is approximately 60% ( $r_{1td}$ ). Case II: If the rate of item response available in the domains approximately 40% ( $r_{2td}$ ). Quantities  $n_{td}$  and  $n_{tdr}$  are computed based on existing information from the populations (quarters).  $n_{td}$  are computed using Yamane's formula  $\left[ \frac{N_{td}}{1 + N_{td} e^2} \right]$ . The data statistics from the population and the sample are shown in Table 2.

#### Estimation of Variances

The estimated MSE and Average MSE for quarters (waves) were carried out with the help of the Minitab software package. The results for the existing and proposed estimators are presented from Tables 3 - 8. The Percentage Relative Efficiency (PRE) was also computed and results are presented from Tables 9-14.

**Table 2:** Data Statistics for IHFCS 2020

| QUARTER | Quantity  | DOMAIN |     |     |     |     |     | TOTALS       |
|---------|-----------|--------|-----|-----|-----|-----|-----|--------------|
|         |           | 1      | 2   | 3   | 4   | 5   | 6   |              |
| $Q_1$   | $N_{td}$  | 447    | 290 | 528 | 220 | 290 | 289 | 2064 ( $N$ ) |
|         | $n_{td}$  | 211    | 168 | 228 | 142 | 168 | 168 | 1085( $n$ )  |
|         | $r_{1td}$ | 127    | 101 | 137 | 85  | 101 | 101 | 652(case1)   |
|         | $r_{2td}$ | 84     | 67  | 91  | 57  | 67  | 67  | 810(case II) |
| $Q_2$   | $N_{td}$  | 449    | 290 | 516 | 211 | 227 | 284 | 1977 ( $N$ ) |
|         | $n_{td}$  | 211    | 168 | 225 | 138 | 145 | 166 | 1053( $n$ )  |
|         | $r_{1td}$ | 127    | 101 | 135 | 83  | 87  | 100 | 633(case1)   |
|         | $r_{2td}$ | 84     | 67  | 90  | 55  | 58  | 66  | 420(case II) |
| $Q_3$   | $N_{td}$  | 449    | 290 | 520 | 210 | 276 | 288 | 2033( $N$ )  |
|         | $n_{td}$  | 211    | 168 | 226 | 138 | 163 | 167 | 1073( $n$ )  |
|         | $r_{1td}$ | 127    | 101 | 136 | 83  | 98  | 100 | 545(case1)   |
|         | $r_{2td}$ | 84     | 67  | 90  | 55  | 65  | 67  | 428(case II) |
| $Q_4$   | $N_{td}$  | 448    | 290 | 522 | 219 | 277 | 288 | 2044 ( $N$ ) |
|         | $n_{td}$  | 211    | 168 | 226 | 142 | 164 | 167 | 1078( $n$ )  |
|         | $r_{1td}$ | 127    | 101 | 136 | 85  | 98  | 100 | 647(case1)   |
|         | $r_{2td}$ | 84     | 67  | 90  | 55  | 67  | 67  | 430(case II) |

 $N = 8138$ **Table 3:** MSE for Scheme 1-Case 1

| $Q_i D_j$ | ESTIMATORS         |                    |                    |                    |
|-----------|--------------------|--------------------|--------------------|--------------------|
|           | $T_{B1}$           | $T_{P1}$           | $T_{DS1}$          | $\tau_{r1}^{SRS}$  |
| $Q_1 D_1$ | 3.72089E+07        | 1.62053E+08        | 3.75653E+07        | 2.90394E+07        |
| $Q_1 D_2$ | 2.03134E+07        | 7.90862E+07        | 2.04139E+07        | 1.97269E+07        |
| $Q_1 D_3$ | 1.26278E+07        | 5.66447E+07        | 1.26936E+07        | 1.07618E+07        |
| $Q_1 D_4$ | 2.45903E+07        | 6.90771E+07        | 2.48443E+07        | 1.10472E+07        |
| $Q_1 D_5$ | 1.30127E+08        | 5.09313E+08        | 1.33390E+08        | 1.14199E+08        |
| $Q_1 D_6$ | 2.14221E+07        | 7.61977E+07        | 2.16094E+07        | 1.48701E+07        |
| AMSE      | <b>69845633</b>    | <b>69845633</b>    | <b>69845633</b>    | <b>69845633</b>    |
| $Q_2 D_1$ | 1.22869E+15        | 1.62052E+15        | 1.25812E+15        | 1.10074E+15        |
| $Q_2 D_2$ | 1.69294E+08        | 5.87751E+08        | 1.76387E+08        | 9.19901E+07        |
| $Q_2 D_3$ | 1.06242E+07        | 4.65129E+07        | 1.06699E+07        | 7.05797E+06        |
| $Q_2 D_4$ | 1.71585E+07        | 4.81762E+07        | 1.73887E+07        | 6.07391E+06        |
| $Q_2 D_5$ | 1.40742E+07        | 4.35649E+07        | 1.41566E+07        | 7.53617E+06        |
| $Q_2 D_6$ | 4.33810E+08        | 1.85628E+09        | 4.99358E+08        | 3.50677E+08        |
| AMSE      | <b>2.04782E+14</b> | <b>2.70087E+14</b> | <b>2.09687E+14</b> | <b>1.83457E+14</b> |
| $Q_3 D_1$ | 2.92295E+07        | 1.22199E+08        | 2.95591E+07        | 1.13542E+07        |
| $Q_3 D_2$ | 1.97761E+07        | 6.61227E+07        | 1.99117E+07        | 1.11172E+07        |
| $Q_3 D_3$ | 1.78111E+07        | 7.60645E+07        | 1.79115E+07        | 1.15817E+07        |
| $Q_3 D_4$ | 1.82363E+07        | 5.12509E+07        | 1.84267E+07        | 8.90449E+06        |
| $Q_3 D_5$ | 1.27236E+07        | 4.41540E+07        | 1.27931E+07        | 8.07717E+06        |
| $Q_3 D_6$ | 6.81919E+06        | 2.27157E+07        | 6.85865E+06        | 3.59790E+06        |
| AMSE      | <b>17432632</b>    | <b>63751133</b>    | <b>17576792</b>    | <b>9105443</b>     |
| $Q_4 D_1$ | 1.77792E+07        | 7.02092E+07        | 1.79378E+07        | 9.86827E+06        |
| $Q_4 D_2$ | 2.44922E+07        | 8.28507E+07        | 2.46597E+07        | 1.36654E+07        |
| $Q_4 D_3$ | 1.53825E+07        | 6.42298E+07        | 1.54611E+07        | 7.49722E+06        |

|           |                 |                 |                 |                 |
|-----------|-----------------|-----------------|-----------------|-----------------|
| $Q_4 D_4$ | 2.04277E+07     | 5.78330E+07     | 2.06079E+07     | 9.92494E+06     |
| $Q_4 D_5$ | 3.25746E+07     | 1.15076E+08     | 3.28623E+07     | 2.40945E+07     |
| $Q_4 D_6$ | 8.66508E+06     | 2.88751E+07     | 8.71365E+06     | 4.81716E+06     |
| AMSE      | <b>19886880</b> | <b>69845633</b> | <b>20040408</b> | <b>11644582</b> |

Source: Statistical computation from original data 2026

**Table 4:** MSE for Scheme 1-Case 2

| $Q_i D_j$ | ESTIMATORS         |                    |                    |                    |
|-----------|--------------------|--------------------|--------------------|--------------------|
|           | $T_{B1}$           | $T_{P1}$           | $T_{DS1}$          | $\tau_{r1}^{SRS}$  |
| $Q_1 D_1$ | 6.56119E+07        | 3.24106E+08        | 6.65139E+07        | 4.49889E+07        |
| $Q_1 D_2$ | 3.59738E+07        | 1.58172E+08        | 3.63108E+07        | 3.33652E+07        |
| $Q_1 D_3$ | 2.21961E+07        | 1.13289E+08        | 2.23692E+07        | 1.61501E+07        |
| $Q_1 D_4$ | 4.92742E+07        | 1.38154E+08        | 5.00673E+07        | 2.01298E+07        |
| $Q_1 D_5$ | 2.29978E+08        | 1.01863E+09        | 2.40007E+08        | 1.99574E+08        |
| $Q_1 D_6$ | 4.07281E+07        |                    | 4.12695E+07        | 2.66792E+07        |
| AMSE      | <b>73960350</b>    | <b>317457667</b>   | <b>76089617</b>    | <b>56814533</b>    |
| $Q_2 D_1$ | 2.07124E+15        | 3.24105E+15        | 2.15617E+15        | 1.87879E+15        |
| $Q_2 D_2$ | 3.22493E+08        | 1.17550E+09        | 3.48924E+08        | 1.60583E+08        |
| $Q_2 D_3$ | 1.88608E+07        | 9.30258E+07        | 1.89766E+07        | 1.35237E+07        |
| $Q_2 D_4$ | 3.50268E+07        | 9.63523E+07        | 3.56826E+07        | 1.42939E+07        |
| $Q_2 D_5$ | 2.74373E+07        | 8.71298E+07        | 2.77007E+07        | 1.58046E+07        |
| $Q_2 D_6$ | 7.23478E+08        | 3.71256E+09        | 9.22710E+08        | 6.10952E+08        |
| AMSE      | <b>3.45207E+14</b> | <b>5.40176E+14</b> | <b>3.59362E+14</b> | <b>3.13132E+14</b> |
| $Q_3 D_1$ | 5.38420E+07        | 2.44397E+08        | 5.46663E+07        | 3.13431E+07        |
| $Q_3 D_2$ | 3.90482E+07        | 1.32245E+08        | 3.94721E+07        | 1.83044E+07        |
| $Q_3 D_3$ | 3.32058E+07        | 1.52129E+08        | 3.34927E+07        | 1.91768E+07        |
| $Q_3 D_4$ | 3.72006E+07        | 1.02502E+08        | 3.78054E+07        | 1.50614E+07        |
| $Q_3 D_5$ | 2.42630E+07        | 8.83079E+07        | 2.44743E+07        | 1.48469E+07        |
| $Q_3 D_6$ | 1.31890E+07        | 4.54313E+07        | 1.32903E+07        | 6.66754E+06        |
| AMSE      | <b>33458100</b>    | <b>127502033</b>   | <b>33866850</b>    | <b>17566690</b>    |
| $Q_4 D_1$ | 3.46104E+07        | 1.40418E+08        | 3.50244E+07        | 1.64370E+07        |
| $Q_4 D_2$ | 4.80249E+07        | 1.65701E+08        | 4.85307E+07        | 2.44539E+07        |
| $Q_4 D_3$ | 2.93609E+07        | 1.28460E+08        | 2.95824E+07        | 1.55310E+07        |
| $Q_4 D_4$ | 4.30525E+07        | 1.15666E+08        | 4.37160E+07        | 1.62906E+07        |
| $Q_4 D_5$ | 5.85979E+07        | 2.30151E+08        | 5.94350E+07        | 4.01856E+07        |
| $Q_4 D_6$ | 1.67500E+07        | 5.77502E+07        | 1.68792E+07        | 8.70244E+06        |
| AMSE      | <b>38399433</b>    | <b>139691033</b>   | <b>38861283</b>    | <b>20266757</b>    |

Source: Statistical computation from original data 2026

**Table 5:** MSE for Scheme 2-Case 1

| $Q_i D_j$ | ESTIMATORS      |                  |                 |                   |
|-----------|-----------------|------------------|-----------------|-------------------|
|           | $T_{B2}$        | $T_{P2}$         | $T_{DS2}$       | $\tau_{r2}^{SRS}$ |
| $Q_1 D_1$ | 3.33881E+07     | 1.50389E+08      | 3.39089E+07     | 2.39089E+07       |
| $Q_1 D_2$ | 2.03316E+07     | 7.90781E+07      | 2.04107E+07     | 1.00107E+07       |
| $Q_1 D_3$ | 1.08831E+07     | 5.07033E+07      | 1.09627E+07     | 1.01627E+07       |
| $Q_1 D_4$ | 1.19523E+07     | 4.23873E+07      | 1.22402E+07     | 1.00402E+07       |
| $Q_1 D_5$ | 1.24428E+08     | 4.95188E+08      | 1.27812E+08     | 1.00812E+08       |
| $Q_1 D_6$ | 1.56682E+07     | 6.17805E+07      | 1.59166E+07     | 1.19166E+07       |
| AMSE      | <b>36108550</b> | <b>146587700</b> | <b>36875183</b> | <b>27808517</b>   |

|           |                    |                    |                    |                    |
|-----------|--------------------|--------------------|--------------------|--------------------|
| $Q_2 D_1$ | 1.22862E+15        | 1.50389E+15        | 1.25812E+15        | 1.00312E+15        |
| $Q_2 D_2$ | 1.02364E+08        | 4.07251E+08        | 1.05115E+08        | 6.05114E+07        |
| $Q_2 D_3$ | 8.72189E+06        | 4.01413E+07        | 8.78200E+06        | 4.78200E+06        |
| $Q_2 D_4$ | 8.53529E+06        | 3.05843E+07        | 8.94146E+06        | 3.94146E+06        |
| $Q_2 D_5$ | 9.36112E+06        | 3.33303E+07        | 9.45110E+06        | 5.45110E+06        |
| $Q_2 D_6$ | 3.90921E+08        | 1.72413E+09        | 4.46818E+08        | 1.46818E+08        |
| AMSE      | <b>2.04770E+14</b> | <b>2.50649E+14</b> | <b>2.09687E+14</b> | <b>1.67187E+14</b> |
| $Q_3 D_1$ | 2.20606E+07        | 9.98492E+07        | 2.25533E+07        | 9.53302E+06        |
| $Q_3 D_2$ | 1.15582E+07        | 4.53195E+07        | 1.16973E+07        | 9.16935E+06        |
| $Q_3 D_3$ | 1.24268E+07        | 5.76749E+07        | 1.25267E+07        | 9.25267E+06        |
| $Q_3 D_4$ | 9.30846E+06        | 3.27612E+07        | 9.54830E+06        | 6.54830E+06        |
| $Q_3 D_5$ | 9.39790E+06        | 3.60095E+07        | 9.47899E+06        | 5.47899E+06        |
| $Q_3 D_6$ | 4.13445E+06        | 1.60959E+07        | 4.20283E+06        | 1.20283E+06        |
| AMSE      | <b>11481068</b>    | <b>47951700</b>    | <b>11667903</b>    | <b>6864193</b>     |
| $Q_4 D_1$ | 1.02476E+07        | 4.63645E+07        | 1.04633E+07        | 8.06332E+06        |
| $Q_4 D_2$ | 1.50092E+07        | 5.89008E+07        | 1.52028E+07        | 1.02021E+07        |
| $Q_4 D_3$ | 9.58070E+06        | 4.44251E+07        | 9.66197E+06        | 4.66197E+06        |
| $Q_4 D_4$ | 1.03180E+07        | 3.63716E+07        | 1.04728E+07        | 8.47283E+06        |
| $Q_4 D_5$ | 2.55509E+07        | 9.79881E+07        | 2.58452E+07        | 2.11452E+07        |
| $Q_4 D_6$ | 5.27467E+06        | 2.04943E+07        | 5.35129E+06        | 2.35129E+06        |
| AMSE      | <b>12663512</b>    | <b>50757400</b>    | <b>12832893</b>    | <b>9149452</b>     |

Source: Statistical computation from original data 2026

Table 6: MSE for Scheme 2-Case 2

| $Q_i D_j$ | ESTIMATORS         |                    |                    |                    |
|-----------|--------------------|--------------------|--------------------|--------------------|
|           | $T_{B2}$           | $T_{P2}$           | $T_{DS2}$          | $\tau_{r2}^{SRS}$  |
| $Q_1 D_1$ | 5.66377E+07        | 3.00777E+08        | 5.81561E+07        | 3.81561E+07        |
| $Q_1 D_2$ | 3.60538E+07        | 1.58156E+08        | 3.63035E+07        | 3.23035E+07        |
| $Q_1 D_3$ | 1.82216E+07        | 1.01407E+08        | 1.84460E+07        | 1.44460E+07        |
| $Q_1 D_4$ | 2.11155E+07        | 8.47746E+07        | 2.20387E+07        | 1.50387E+07        |
| $Q_1 D_5$ | 2.16841E+08        | 9.90376E+08        | 2.27333E+08        | 1.27333E+08        |
| $Q_1 D_6$ | 2.75528E+07        | 1.23561E+08        | 2.83330E+07        | 2.23330E+07        |
| AMSE      | <b>62737067</b>    | <b>293175267</b>   | <b>65101717</b>    | <b>41601717</b>    |
| $Q_2 D_1$ | 2.07096E+15        | 3.00778E+15        | 2.15616E+15        | 1.35616E+15        |
| $Q_2 D_2$ | 1.78434E+08        | 8.14502E+08        | 1.86962E+08        | 1.46962E+08        |
| $Q_2 D_3$ | 1.45604E+07        | 8.02827E+07        | 1.47289E+07        | 1.17289E+07        |
| $Q_2 D_4$ | 1.50941E+07        | 6.11687E+07        | 1.64452E+07        | 1.04452E+07        |
| $Q_2 D_5$ | 1.68200E+07        | 6.66606E+07        | 1.71132E+07        | 1.21132E+07        |
| $Q_2 D_6$ | 6.38238E+08        | 3.44827E+09        | 8.02094E+08        | 5.02094E+08        |
| AMSE      | <b>3.45160E+14</b> | <b>5.01297E+14</b> | <b>3.59360E+14</b> | <b>2.26027E+14</b> |
| $Q_3 D_1$ | 3.72199E+07        | 1.99698E+08        | 3.86520E+07        | 1.86520E+07        |
| $Q_3 D_2$ | 2.03682E+07        | 9.06390E+07        | 2.08054E+07        | 1.08054E+07        |
| $Q_3 D_3$ | 2.09120E+07        | 1.15350E+08        | 2.11967E+07        | 1.11967E+07        |
| $Q_3 D_4$ | 1.67822E+07        | 6.55224E+07        | 1.75861E+07        | 1.15861E+07        |
| $Q_3 D_5$ | 1.66833E+07        | 7.20190E+07        | 1.69409E+07        | 1.09409E+07        |
| $Q_3 D_6$ | 7.16480E+06        | 3.21919E+07        | 7.37397E+06        | 4.37397E+06        |
| AMSE      | <b>19855067</b>    | <b>95903383</b>    | <b>20425845</b>    | <b>11259178</b>    |
| $Q_4 D_1$ | 1.73098E+07        | 9.27291E+07        | 1.79387E+07        | 1.19387E+07        |
| $Q_4 D_2$ | 2.64321E+07        | 1.17802E+08        | 2.70404E+07        | 2.00404E+07        |

|             |                 |                  |                 |                 |
|-------------|-----------------|------------------|-----------------|-----------------|
| $Q_4 D_3$   | 1.61087E+07     | 8.88501E+07      | 1.63403E+07     | 1.03403E+07     |
| $Q_4 D_4$   | 1.92600E+07     | 7.27432E+07      | 1.98089E+07     | 1.18089E+07     |
| $Q_4 D_5$   | 4.34900E+07     | 1.95976E+08      | 4.43505E+07     | 3.43505E+07     |
| $Q_4 D_6$   | 9.15450E+06     | 4.09886E+07      | 9.38898E+06     | 7.38898E+06     |
| <b>AMSE</b> | <b>21959183</b> | <b>101514833</b> | <b>22477963</b> | <b>15977963</b> |

Source: Statistical computation from original data 2026

**Table 7: MSE for Scheme 3-Case 1**

| $Q_i D_j$   | ESTIMATORS         |                    |                    |                    |
|-------------|--------------------|--------------------|--------------------|--------------------|
|             | $T_{B3}$           | $T_{P3}$           | $T_{DS3}$          | $\tau_{r3}^{SRS}$  |
| $Q_1 D_1$   | 3.64405E+07        | 1.67885E+08        | 3.68275E+07        | 2.89423E+07        |
| $Q_1 D_2$   | 2.03203E+07        | 7.90903E+07        | 2.04128E+07        | 1.74073E+07        |
| $Q_1 D_3$   | 1.23756E+07        | 5.96155E+07        | 1.24433E+07        | 1.03703E+07        |
| $Q_1 D_4$   | 1.86344E+07        | 8.24220E+07        | 1.89041E+07        | 1.05576E+07        |
| $Q_1 D_5$   | 1.28042E+08        | 5.16375E+08        | 1.31349E+08        | 1.01228E+08        |
| $Q_1 D_6$   | 1.93008E+07        | 8.34063E+07        | 1.95096E+07        | 1.35676E+07        |
| <b>AMSE</b> | <b>39185600</b>    | <b>164799017</b>   | <b>39907717</b>    | <b>30345517</b>    |
| $Q_2 D_1$   | 1.22868E+15        | 1.67823E+15        | 1.25812E+15        | 1.08435E+15        |
| $Q_2 D_2$   | 1.45036E+08        | 6.78000E+08        | 1.50314E+08        | 8.14476E+07        |
| $Q_2 D_3$   | 1.03312E+07        | 4.96987E+07        | 1.03790E+07        | 6.28402E+06        |
| $Q_2 D_4$   | 1.30435E+07        | 5.69721E+07        | 1.33518E+07        | 5.45266E+06        |
| $Q_2 D_5$   | 1.19149E+07        | 4.86822E+07        | 1.20008E+07        | 6.08090E+06        |
| $Q_2 D_6$   | 4.18003E+08        | 1.92235E+09        | 4.79894E+08        | 2.49247E+08        |
| <b>AMSE</b> | <b>2.04780E+14</b> | <b>2.79705E+14</b> | <b>2.09687E+14</b> | <b>1.80725E+14</b> |
| $Q_3 D_1$   | 2.78084E+07        | 1.33373E+08        | 2.81679E+07        | 1.04290E+07        |
| $Q_3 D_2$   | 1.67698E+07        | 7.65243E+07        | 1.69067E+07        | 9.82394E+06        |
| $Q_3 D_3$   | 1.70269E+07        | 8.52593E+07        | 1.71272E+07        | 1.00416E+07        |
| $Q_3 D_4$   | 1.39284E+07        | 6.04957E+07        | 1.41420E+07        | 7.83701E+06        |
| $Q_3 D_5$   | 1.14509E+07        | 4.82262E+07        | 1.15247E+07        | 7.08279E+06        |
| $Q_3 D_6$   | 5.81890E+06        | 2.60255E+07        | 5.86822E+06        | 2.46715E+06        |
| <b>AMSE</b> | <b>15467217</b>    | <b>71650667</b>    | <b>15622787</b>    | <b>7946915</b>     |
| $Q_4 D_1$   | 1.62723E+07        | 8.21315E+07        | 1.64416E+07        | 8.79863E+06        |
| $Q_4 D_2$   | 2.10233E+07        | 9.48257E+07        | 2.12001E+07        | 1.16701E+07        |
| $Q_4 D_3$   | 1.45522E+07        | 7.41322E+07        | 1.46311E+07        | 5.23455E+06        |
| $Q_4 D_4$   | 1.56189E+07        | 6.85637E+07        | 1.57868E+07        | 8.82447E+06        |
| $Q_4 D_5$   | 2.98054E+07        | 1.23619E+08        | 3.00957E+07        | 2.25077E+07        |
| $Q_4 D_6$   | 7.40144E+06        | 3.30655E+07        | 7.45973E+06        | 2.93293E+06        |
| <b>AMSE</b> | <b>17445590</b>    | <b>79389600</b>    | <b>17602505</b>    | <b>9994730</b>     |

Source: Statistical computation from original data 2026

**Table 8: MSE for Scheme 3-Case 2**

| $Q_i D_j$ | ESTIMATORS  |             |             |                   |
|-----------|-------------|-------------|-------------|-------------------|
|           | $T_{B3}$    | $T_{P3}$    | $T_{DS3}$   | $\tau_{r3}^{SRS}$ |
| $Q_1 D_1$ | 5.97890E+07 | 3.24106E+08 | 6.10747E+07 | 3.91895E+07       |
| $Q_1 D_2$ | 3.60327E+07 | 1.58172E+08 | 3.63055E+07 | 3.27300E+07       |
| $Q_1 D_3$ | 1.97224E+07 | 1.13289E+08 | 1.99266E+07 | 1.53536E+07       |
| $Q_1 D_4$ | 2.78111E+07 | 1.38154E+08 | 2.87026E+07 | 1.80056E+07       |
| $Q_1 D_5$ | 2.20508E+08 | 1.01863E+09 | 2.30870E+08 | 1.60749E+08       |
| $Q_1 D_6$ | 3.12163E+07 | 1.52395E+08 | 3.19260E+07 | 2.59840E+07       |

|           |                    |                    |                    |                    |
|-----------|--------------------|--------------------|--------------------|--------------------|
| AMSE      | <b>65846583</b>    | <b>317457667</b>   | <b>68134233</b>    | <b>48668617</b>    |
| $Q_2 D_1$ | 2.07106E+15        | 3.24105E+15        | 2.15617E+15        | 1.58240E+15        |
| $Q_2 D_2$ | 2.19579E+08        | 1.17550E+09        | 2.32161E+08        | 1.56345E+08        |
| $Q_2 D_3$ | 1.61784E+07        | 9.30258E+07        | 1.63259E+07        | 1.27874E+07        |
| $Q_2 D_4$ | 1.96845E+07        | 9.63523E+07        | 2.08555E+07        | 1.29564E+07        |
| $Q_2 D_5$ | 1.93770E+07        | 8.71298E+07        | 1.96629E+07        | 1.42430E+07        |
| $Q_2 D_6$ | 6.61906E+08        | 3.71256E+09        | 8.35170E+08        | 6.04523E+08        |
| AMSE      | <b>3.45177E+14</b> | <b>5.40176E+14</b> | <b>3.59362E+14</b> | <b>2.63733E+14</b> |
| $Q_3 D_1$ | 4.30657E+07        | 2.44397E+08        | 4.42665E+07        | 2.85277E+07        |
| $Q_3 D_2$ | 2.55812E+07        | 1.32245E+08        | 2.60148E+07        | 1.23400E+07        |
| $Q_3 D_3$ | 2.55117E+07        | 1.52129E+08        | 2.57973E+07        | 1.23117E+07        |
| $Q_3 D_4$ | 2.14235E+07        | 1.02502E+08        | 2.21798E+07        | 1.48748E+07        |
| $Q_3 D_5$ | 1.87421E+07        | 8.83079E+07        | 1.89866E+07        | 1.25447E+07        |
| $Q_3 D_6$ | 8.86440E+06        | 4.54313E+07        | 9.03936E+06        | 5.26389E+06        |
| AMSE      | <b>23864767</b>    | <b>127502033</b>   | <b>24380727</b>    | <b>14310465</b>    |
| $Q_4 D_1$ | 2.33682E+07        | 1.40418E+08        | 2.39170E+07        | 1.54610E+07        |
| $Q_4 D_2$ | 3.24589E+07        | 1.65701E+08        | 3.30377E+07        | 2.25047E+07        |
| $Q_4 D_3$ | 2.10817E+07        | 1.28460E+08        | 2.13094E+07        | 1.39128E+07        |
| $Q_4 D_4$ | 2.45494E+07        | 1.15666E+08        | 2.51228E+07        | 1.33605E+07        |
| $Q_4 D_5$ | 4.77471E+07        | 2.30151E+08        | 4.86009E+07        | 3.77129E+07        |
| $Q_4 D_6$ | 1.12957E+07        | 5.77502E+07        | 1.14974E+07        | 8.47062E+06        |
| AMSE      | <b>26750167</b>    | <b>139691033</b>   | <b>27247533</b>    | <b>18570420</b>    |

Source: Statistical computation from original data 2026

**Table 9:** PRE for Scheme 1-Case 1

| $Q_i D_j$ | ESTIMATORS |              |               |                   |
|-----------|------------|--------------|---------------|-------------------|
|           | $T_{B1}$   | $T_{P1}$     | $T_{DS1}$     | $\tau_{r1}^{SRS}$ |
| $Q_1 D_1$ | 100        | 22.96        | 99.05         | 128.13            |
| $Q_1 D_2$ | 100        | 25.69        | 99.51         | 102.97            |
| $Q_1 D_3$ | 100        | 22.29        | 99.48         | 117.34            |
| $Q_1 D_4$ | 100        | 35.6         | 98.98         | 222.59            |
| $Q_1 D_5$ | 100        | 25.55        | 97.55         | 113.95            |
| $Q_1 D_6$ | 100        | 28.11        | 99.13         | 144.06            |
| APRE      | <b>100</b> | <b>26.70</b> | <b>98.950</b> | <b>138.2</b>      |
| $Q_2 D_1$ | 100        | 75.82        | 97.66         | 111.66            |
| $Q_2 D_2$ | 100        | 28.8         | 95.98         | 184.04            |
| $Q_2 D_3$ | 100        | 22.84        | 99.57         | 150.53            |
| $Q_2 D_4$ | 100        | 35.62        | 98.68         | 282.5             |
| $Q_2 D_5$ | 100        | 32.31        | 99.42         | 186.76            |
| $Q_2 D_6$ | 100        | 23.37        | 86.87         | 123.71            |
| APRE      | <b>100</b> | <b>36.46</b> | <b>96.36</b>  | <b>173.2</b>      |
| $Q_3 D_1$ | 100        | 23.92        | 98.88         | 257.43            |
| $Q_3 D_2$ | 100        | 29.91        | 99.32         | 177.89            |
| $Q_3 D_3$ | 100        | 23.42        | 99.44         | 153.79            |
| $Q_3 D_4$ | 100        | 35.58        | 98.97         | 204.8             |
| $Q_3 D_5$ | 100        | 28.82        | 99.46         | 157.53            |
| $Q_3 D_6$ | 100        | 30.02        | 99.42         | 189.53            |
| APRE      | <b>100</b> | <b>28.61</b> | <b>99.248</b> | <b>190.2</b>      |

|           |            |              |               |              |
|-----------|------------|--------------|---------------|--------------|
| $Q_4 D_1$ | 100        | 25.32        | 99.12         | 180.17       |
| $Q_4 D_2$ | 100        | 29.56        | 99.32         | 179.23       |
| $Q_4 D_3$ | 100        | 23.95        | 99.49         | 205.18       |
| $Q_4 D_4$ | 100        | 35.32        | 99.13         | 205.82       |
| $Q_4 D_5$ | 100        | 28.31        | 99.12         | 135.2        |
| $Q_4 D_6$ | 100        | 30.01        | 99.44         | 179.88       |
| APRE      | <b>100</b> | <b>28.75</b> | <b>99.270</b> | <b>180.9</b> |

Source: Statistical computation from original data 2026

**Table 10:** PRE for Scheme 1-Case 2

| $Q_i D_j$ | ESTIMATORS |              |               |                   |
|-----------|------------|--------------|---------------|-------------------|
|           | $T_{B1}$   | $T_{P1}$     | $T_{DS1}$     | $\tau_{r1}^{SRS}$ |
| $Q_1 D_1$ | 100        | 20.24        | 98.64         | 145.84            |
| $Q_1 D_2$ | 100        | 22.74        | 99.07         | 107.82            |
| $Q_1 D_3$ | 100        | 19.59        | 99.23         | 137.44            |
| $Q_1 D_4$ | 100        | 35.67        | 98.42         | 244.78            |
| $Q_1 D_5$ | 100        | 22.58        | 95.82         | 115.23            |
| $Q_1 D_6$ | 100        | 26.73        | 98.69         | 152.66            |
| APRE      | <b>100</b> | <b>24.59</b> | <b>98.312</b> | <b>150.6</b>      |
| $Q_2 D_1$ | 100        | 63.91        | 96.06         | 110.24            |
| $Q_2 D_2$ | 100        | 27.43        | 92.42         | 200.83            |
| $Q_2 D_3$ | 100        | 20.27        | 99.39         | 139.46            |
| $Q_2 D_4$ | 100        | 36.35        | 98.16         | 245.05            |
| $Q_2 D_5$ | 100        | 31.49        | 99.05         | 173.6             |
| $Q_2 D_6$ | 100        | 19.49        | 78.41         | 118.42            |
| APRE      | <b>100</b> | <b>33.16</b> | <b>93.92</b>  | <b>164.6</b>      |
| $Q_3 D_1$ | 100        | 22.03        | 98.49         | 171.78            |
| $Q_3 D_2$ | 100        | 29.53        | 98.93         | 213.33            |
| $Q_3 D_3$ | 100        | 21.83        | 99.14         | 173.16            |
| $Q_3 D_4$ | 100        | 36.29        | 98.4          | 246.99            |
| $Q_3 D_5$ | 100        | 27.48        | 99.14         | 163.42            |
| $Q_3 D_6$ | 100        | 29.03        | 99.24         | 197.81            |
| APRE      | <b>100</b> | <b>27.70</b> | <b>98.890</b> | <b>194.4</b>      |
| $Q_4 D_1$ | 100        | 24.65        | 98.82         | 210.56            |
| $Q_4 D_2$ | 100        | 28.98        | 98.96         | 196.39            |
| $Q_4 D_3$ | 100        | 22.86        | 99.25         | 189.05            |
| $Q_4 D_4$ | 100        | 37.22        | 98.48         | 264.28            |
| $Q_4 D_5$ | 100        | 25.46        | 98.59         | 145.82            |
| $Q_4 D_6$ | 100        | 29           | 99.23         | 192.47            |
| APRE      | <b>100</b> | <b>28.03</b> | <b>98.888</b> | <b>199.8</b>      |

Source: Statistical computation from original data 2026

**Table 11:** PRE for Scheme 2 Case 1

| $Q_i D_j$ | ESTIMATORS |              |               |                   |
|-----------|------------|--------------|---------------|-------------------|
|           | $T_{B2}$   | $T_{P2}$     | $T_{DS2}$     | $\tau_{r2}^{SRS}$ |
| $Q_1 D_1$ | 100        | 22.2         | 98.46         | 139.65            |
| $Q_1 D_2$ | 100        | 25.71        | 99.61         | 203.1             |
| $Q_1 D_3$ | 100        | 21.46        | 99.27         | 107.09            |
| $Q_1 D_4$ | 100        | 28.2         | 97.65         | 119.04            |
| $Q_1 D_5$ | 100        | 25.13        | 97.35         | 123.43            |
| $Q_1 D_6$ | 100        | 25.36        | 98.44         | 131.48            |
| APRE      | <b>100</b> | <b>24.68</b> | <b>98.463</b> | <b>137.3</b>      |
| $Q_2 D_1$ | 100        | 81.7         | 97.66         | 122.48            |
| $Q_2 D_2$ | 100        | 25.14        | 97.38         | 169.17            |
| $Q_2 D_3$ | 100        | 21.73        | 99.32         | 182.39            |

|           |            |              |               |              |
|-----------|------------|--------------|---------------|--------------|
| $Q_2 D_4$ | 100        | 27.91        | 95.46         | 216.55       |
| $Q_2 D_5$ | 100        | 28.09        | 99.05         | 171.73       |
| $Q_2 D_6$ | 100        | 22.67        | 87.49         | 266.26       |
| APRE      | <b>100</b> | <b>34.54</b> | <b>96.06</b>  | <b>188.1</b> |
| $Q_3 D_1$ | 100        | 22.09        | 97.82         | 231.41       |
| $Q_3 D_2$ | 100        | 25.5         | 98.81         | 126.05       |
| $Q_3 D_3$ | 100        | 21.55        | 99.2          | 134.3        |
| $Q_3 D_4$ | 100        | 28.41        | 97.49         | 142.15       |
| $Q_3 D_5$ | 100        | 26.1         | 99.14         | 171.53       |
| $Q_3 D_6$ | 100        | 25.69        | 98.37         | 343.73       |
| APRE      | <b>100</b> | <b>24.89</b> | <b>98.472</b> | <b>191.5</b> |
| $Q_4 D_1$ | 100        | 22.1         | 97.94         | 127.09       |
| $Q_4 D_2$ | 100        | 25.48        | 98.73         | 147.12       |
| $Q_4 D_3$ | 100        | 21.57        | 99.16         | 205.51       |
| $Q_4 D_4$ | 100        | 28.37        | 98.52         | 121.78       |
| $Q_4 D_5$ | 100        | 26.08        | 98.86         | 120.84       |
| $Q_4 D_6$ | 100        | 25.74        | 98.57         | 224.33       |
| APRE      | <b>100</b> | <b>24.89</b> | <b>98.630</b> | <b>157.8</b> |

Source: Statistical computation from original data 2026

Table 12: PRE for Scheme 2-Case 2

| $Q_i D_j$ | ESTIMATORS |              |               |                   |
|-----------|------------|--------------|---------------|-------------------|
|           | $T_{B2}$   | $T_{P2}$     | $T_{DS2}$     | $\tau_{r2}^{SRS}$ |
| $Q_1 D_1$ | 100        | 18.83        | 97.39         | 148.44            |
| $Q_1 D_2$ | 100        | 22.8         | 99.31         | 111.61            |
| $Q_1 D_3$ | 100        | 17.97        | 98.78         | 126.14            |
| $Q_1 D_4$ | 100        | 24.91        | 95.81         | 140.41            |
| $Q_1 D_5$ | 100        | 21.89        | 95.38         | 170.29            |
| $Q_1 D_6$ | 100        | 22.3         | 97.25         | 123.37            |
| APRE      | <b>100</b> | <b>21.45</b> | <b>97.320</b> | <b>136.71</b>     |
| $Q_2 D_1$ | 100        | 68.85        | 96.05         | 152.71            |
| $Q_2 D_2$ | 100        | 21.91        | 95.44         | 121.41            |
| $Q_2 D_3$ | 100        | 18.14        | 98.86         | 124.14            |
| $Q_2 D_4$ | 100        | 24.68        | 91.78         | 144.51            |
| $Q_2 D_5$ | 100        | 25.23        | 98.29         | 138.86            |
| $Q_2 D_6$ | 100        | 18.51        | 79.57         | 127.12            |
| APRE      | <b>100</b> | <b>29.55</b> | <b>93.33</b>  | <b>134.79</b>     |
| $Q_3 D_1$ | 100        | 18.64        | 96.29         | 199.55            |
| $Q_3 D_2$ | 100        | 22.47        | 97.9          | 188.5             |
| $Q_3 D_3$ | 100        | 18.13        | 98.66         | 186.77            |
| $Q_3 D_4$ | 100        | 25.61        | 95.43         | 144.85            |
| $Q_3 D_5$ | 100        | 23.17        | 98.48         | 152.49            |
| $Q_3 D_6$ | 100        | 22.26        | 97.16         | 163.81            |
| APRE      | <b>100</b> | <b>21.71</b> | <b>97.320</b> | <b>172.66</b>     |
| $Q_4 D_1$ | 100        | 18.67        | 96.49         | 144.99            |
| $Q_4 D_2$ | 100        | 22.44        | 97.75         | 131.89            |
| $Q_4 D_3$ | 100        | 18.13        | 98.58         | 155.79            |
| $Q_4 D_4$ | 100        | 26.48        | 97.23         | 163.1             |
| $Q_4 D_5$ | 100        | 22.19        | 98.06         | 126.61            |
| $Q_4 D_6$ | 100        | 22.33        | 97.5          | 123.89            |
| APRE      | <b>100</b> | <b>21.71</b> | <b>97.602</b> | <b>141.04</b>     |

Source: Statistical computation from original data 2026

**Table 13: PRE for Scheme 3 Case 1**

| $Q_i D_j$ | ESTIMATORS |               |               |                   |
|-----------|------------|---------------|---------------|-------------------|
|           | $T_{B3}$   | $T_{P3}$      | $T_{DS3}$     | $\tau_{r3}^{SRS}$ |
| $Q_1 D_1$ | 100        | 21.71         | 98.95         | 125.91            |
| $Q_1 D_2$ | 100        | 25.69         | 99.55         | 116.73            |
| $Q_1 D_3$ | 100        | 20.76         | 99.46         | 119.34            |
| $Q_1 D_4$ | 100        | 22.61         | 98.57         | 176.5             |
| $Q_1 D_5$ | 100        | 24.8          | 97.48         | 126.49            |
| $Q_1 D_6$ | 100        | 23.14         | 98.93         | 142.26            |
| APRE      | <b>100</b> | <b>23.118</b> | <b>98.823</b> | <b>134.54</b>     |
| $Q_2 D_1$ | 100        | 73.21         | 97.66         | 113.31            |
| $Q_2 D_2$ | 100        | 21.39         | 96.49         | 178.07            |
| $Q_2 D_3$ | 100        | 20.79         | 99.54         | 164.4             |
| $Q_2 D_4$ | 100        | 22.89         | 97.69         | 239.21            |
| $Q_2 D_5$ | 100        | 24.47         | 99.28         | 195.94            |
| $Q_2 D_6$ | 100        | 21.74         | 87.1          | 167.71            |
| APRE      | <b>100</b> | <b>30.75</b>  | <b>96.29</b>  | <b>176.4</b>      |
| $Q_3 D_1$ | 100        | 20.85         | 98.72         | 266.65            |
| $Q_3 D_2$ | 100        | 21.91         | 99.19         | 170.7             |
| $Q_3 D_3$ | 100        | 19.97         | 99.41         | 169.56            |
| $Q_3 D_4$ | 100        | 23.02         | 98.49         | 177.73            |
| $Q_3 D_5$ | 100        | 23.74         | 99.36         | 161.67            |
| $Q_3 D_6$ | 100        | 22.36         | 99.16         | 235.85            |
| APRE      | <b>100</b> | <b>21.975</b> | <b>99.055</b> | <b>197.0</b>      |
| $Q_4 D_1$ | 100        | 19.81         | 98.97         | 184.94            |
| $Q_4 D_2$ | 100        | 22.17         | 99.17         | 180.15            |
| $Q_4 D_3$ | 100        | 19.63         | 99.46         | 278               |
| $Q_4 D_4$ | 100        | 22.78         | 98.94         | 177               |
| $Q_4 D_5$ | 100        | 24.11         | 99.04         | 132.42            |
| $Q_4 D_6$ | 100        | 22.38         | 99.22         | 252.36            |
| APRE      | <b>100</b> | <b>21.813</b> | <b>99.133</b> | <b>200.8</b>      |

Source: Statistical computation from original data 2026

**Table 14: PRE for Scheme 3 Case 2**

| $Q_i D_j$ | ESTIMATORS |              |               |                   |
|-----------|------------|--------------|---------------|-------------------|
|           | $T_{B3}$   | $T_{P3}$     | $T_{DS3}$     | $\tau_{r3}^{SRS}$ |
| $Q_1 D_1$ | 100        | 17.81        | 97.89         | 152.56            |
| $Q_1 D_2$ | 100        | 22.78        | 99.25         | 110.09            |
| $Q_1 D_3$ | 100        | 16.54        | 98.98         | 128.45            |
| $Q_1 D_4$ | 100        | 16.87        | 96.89         | 154.46            |
| $Q_1 D_5$ | 100        | 21.35        | 95.51         | 137.18            |
| $Q_1 D_6$ | 100        | 18.71        | 97.78         | 120.14            |
| APRE      | <b>100</b> | <b>19.01</b> | <b>97.717</b> | <b>133.81</b>     |
| $Q_2 D_1$ | 100        | 61.7         | 96.05         | 130.88            |
| $Q_2 D_2$ | 100        | 16.19        | 94.58         | 140.45            |
| $Q_2 D_3$ | 100        | 16.28        | 99.1          | 126.52            |
| $Q_2 D_4$ | 100        | 17.28        | 94.39         | 151.93            |
| $Q_2 D_5$ | 100        | 19.9         | 98.55         | 136.05            |
| $Q_2 D_6$ | 100        | 17.22        | 79.25         | 109.49            |
| APRE      | <b>100</b> | <b>24.76</b> | <b>93.65</b>  | <b>132.55</b>     |
| $Q_3 D_1$ | 100        | 16.14        | 97.29         | 150.96            |

|           |            |               |               |               |
|-----------|------------|---------------|---------------|---------------|
| $Q_3 D_2$ | 100        | 16.71         | 98.33         | 207.3         |
| $Q_3 D_3$ | 100        | 14.96         | 98.89         | 207.21        |
| $Q_3 D_4$ | 100        | 17.71         | 96.59         | 144.03        |
| $Q_3 D_5$ | 100        | 19.43         | 98.71         | 149.4         |
| $Q_3 D_6$ | 100        | 17.03         | 98.06         | 168.4         |
| APRE      | <b>100</b> | <b>16.997</b> | <b>97.978</b> | <b>171.2</b>  |
| $Q_4 D_1$ | 100        | 14.23         | 97.71         | 151.14        |
| $Q_4 D_2$ | 100        | 17.12         | 98.25         | 144.23        |
| $Q_4 D_3$ | 100        | 14.22         | 98.93         | 151.53        |
| $Q_4 D_4$ | 100        | 17.9          | 97.72         | 183.75        |
| $Q_4 D_5$ | 100        | 19.31         | 98.24         | 126.61        |
| $Q_4 D_6$ | 100        | 17.08         | 98.25         | 133.35        |
| APRE      | <b>100</b> | <b>16.643</b> | <b>98.183</b> | <b>148.44</b> |

Source: Statistical computation from original data 2026

### Estimation of Domains Means

The population means estimate for the existing and proposed estimators were computed using real-life dataset from the four quarters. Both cases of item response rate were considered. The results for both cases are displayed in Table 15 while domain estimates for each quarter and domain are presented in Table 16.

**Table 15:** Imputed mean value for each scheme

| $Q_i D_i$ | Imputed value for case 1 |                     |                     | Imputed value for case 2 |                     |                     |
|-----------|--------------------------|---------------------|---------------------|--------------------------|---------------------|---------------------|
|           | $(\tau_{r1}^{SRS})$      | $(\tau_{r2}^{SRS})$ | $(\tau_{r3}^{SRS})$ | $(\tau_{r1}^{SRS})$      | $(\tau_{r2}^{SRS})$ | $(\tau_{r3}^{SRS})$ |
| $Q_1 D_1$ | 76202                    | 74780               | 73578               | 117835                   | 101982              | 94755               |
| $Q_1 D_2$ | 78683                    | 68683               | 154903              | 102543                   | 48683               | 155966              |
| $Q_1 D_3$ | 74313                    | 69861               | 64366               | 86918                    | 80430               | 72783               |
| $Q_1 D_4$ | 69367                    | 71617               | 79152               | 82400                    | 83530               | 87723               |
| $Q_1 D_5$ | 80588                    | 82930               | 88397               | 43754                    | 42078               | 39486               |
| $Q_1 D_6$ | 140941                   | 129580              | 108710              | 62997                    | 55277               | 43274               |
| AE        | <b>63433</b>             | <b>82908</b>        | <b>94850</b>        | <b>82741</b>             | <b>68663</b>        | <b>82331</b>        |
| $Q_2 D_1$ | 61022                    | 64224               | 68494               | 69106                    | 73298               | 78087               |
| $Q_2 D_2$ | 65092                    | 70771               | 45005               | 58016                    | 58605               | 57672               |
| $Q_2 D_3$ | 68403                    | 65534               | 62912               | 69501                    | 67165               | 65059               |
| $Q_2 D_4$ | 60573                    | 64074               | 70456               | 62360                    | 65686               | 71914               |
| $Q_2 D_5$ | 51565                    | 51267               | 54292               | 44897                    | 40897               | 38573               |
| $Q_2 D_6$ | 143570                   | 150293              | 101808              | 57701                    | 66305               | 21724               |
| AE        | <b>75038</b>             | <b>77694</b>        | <b>67161</b>        | <b>60264</b>             | <b>61993</b>        | <b>55505</b>        |
| $Q_3 D_1$ | 88423                    | 81811               | 74439               | 93403                    | 92583               | 90527               |
| $Q_3 D_2$ | 73225                    | 72492               | 70025               | 69431                    | 71291               | 78944               |
| $Q_3 D_3$ | 70641                    | 69988               | 66520               | 88382                    | 83206               | 64592               |
| $Q_3 D_4$ | 53977                    | 55569               | 59747               | 47812                    | 51014               | 60300               |
| $Q_3 D_5$ | 58964                    | 59908               | 62708               | 42411                    | 41465               | 40284               |
| $Q_3 D_6$ | 66282                    | 57707               | 46640               | 63580                    | 51893               | 39450               |
| AE        | <b>68585</b>             | <b>66246</b>        | <b>63347</b>        | <b>67503</b>             | <b>65242</b>        | <b>62350</b>        |
| $Q_4 D_1$ | 74030                    | 71743               | 66438               | 79321                    | 79859               | 79712               |
| $Q_4 D_2$ | 90963                    | 90956               | 91020               | 86516                    | 89799               | 99618               |
| $Q_4 D_3$ | 71935                    | 71047               | 67004               | 86518                    | 83903               | 73990               |
| $Q_4 D_4$ | 57130                    | 57966               | 63629               | 52848                    | 54340               | 65291               |
| $Q_4 D_5$ | 79197                    | 80413               | 85521               | 68877                    | 66599               | 62634               |
| $Q_4 D_6$ | 65796                    | 59426               | 48153               | 62753                    | 54529               | 41828               |
| AE        | <b>73175</b>             | <b>71925</b>        | <b>70294</b>        | <b>72806</b>             | <b>71505</b>        | <b>70512</b>        |

Source: Statistical computation from original data 2026

**Table 16:** Domain Estimate using Scheme 2 Case 1

| Quarter | Domain Estimate for Single Stage |              |              |              |              |              | Total        |
|---------|----------------------------------|--------------|--------------|--------------|--------------|--------------|--------------|
|         | D1                               | D2           | D3           | D4           | D5           | D6           |              |
| Q1      | 74780                            | 68683        | 69861        | 71617        | 82930        | 129580       | <b>82908</b> |
| Q2      | 64224                            | 70771        | 65534        | 72074        | 51267        | 150293       | <b>79027</b> |
| Q3      | 81811                            | 72492        | 69988        | 55569        | 59908        | 57707        | <b>66246</b> |
| Q4      | 71743                            | 90956        | 71047        | 87966        | 80413        | 59426        | <b>71925</b> |
| Total   | <b>73149</b>                     | <b>75726</b> | <b>69108</b> | <b>64307</b> | <b>62307</b> | <b>99252</b> |              |

Source: Statistical computation from original data 2026

### Visualization of the Performance of Estimators Across the Waves in 2020

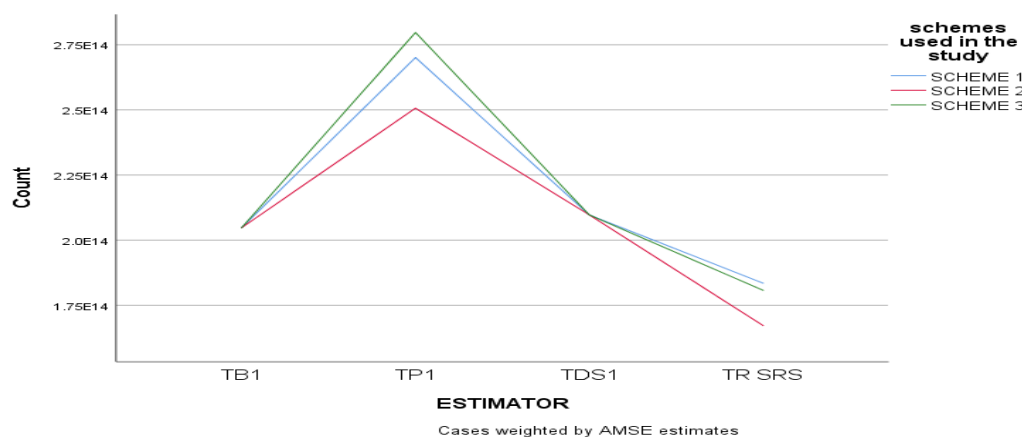
This section deals with the visualization of the average performance across the quarters of 2020 of both existing and proposed estimators in terms of average mean squared error (AMSE), average percentage relative efficiency (APRE), and average mean estimate (AME). Both cases of item response rate were considered. The results for both cases are displayed in Figures 1-5.

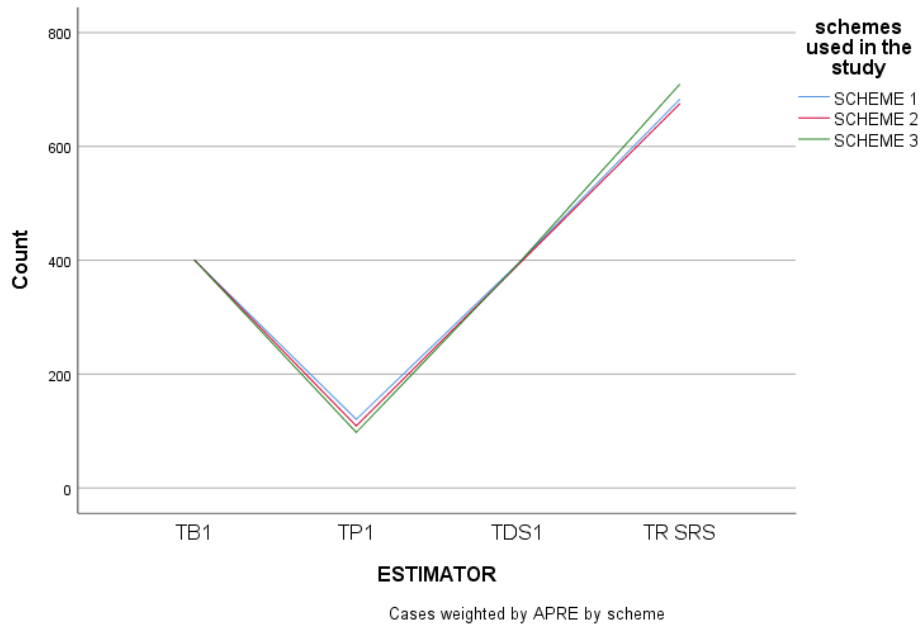
### 5.0 Discussion

This study considers theoretical formulations, analytical comparisons, and empirical investigations to justify the performance of the proposed estimators compared to existing ones. From the results of the empirical analysis, it is observed that Quarter 2, Domain 1 has the highest average expenditure as well as the highest variance. This can be attributed to the low income recorded as reported by the respondents.

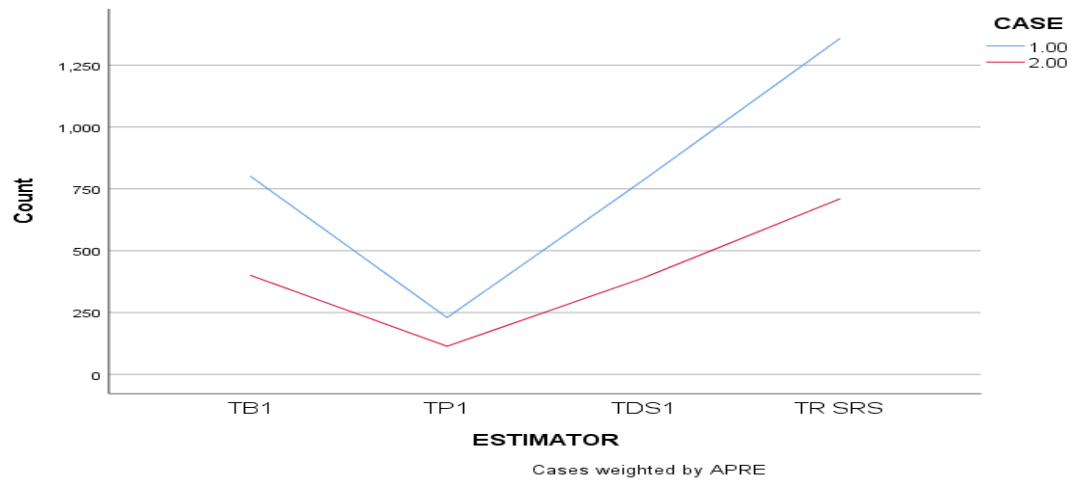
The estimated MSE for Case I and Case II revealed that the proposed logarithmic imputation estimators ( $\tau_{ri}^{SR5}$ ) have lower MSE values than the classical and existing estimators ( $T_{Bi}$ ,  $T_{Pi}$  and  $T_{DSi}$ ) across the six domains in the four quarters as seen in Tables 3 - 8. This implies that the proposed logarithmic imputation estimators are more efficient than the classical estimator (Bhushan et al., 2023) and the existing estimators (Pandy et al., 2021; Das & Singh, 2023).

The performance efficiency of the estimators was considered under two scenarios; when the response item rate was 60% and when it was 40%. It was observed that both the existing and the proposed estimators gained more efficiency in Case I than in Case II which is an indication of the negative effect of item nonresponse. This supports Peytchew's (2012) argument that nonresponse in survey sampling reduces trust in survey estimates due to increased MSE, which makes estimators less efficient.

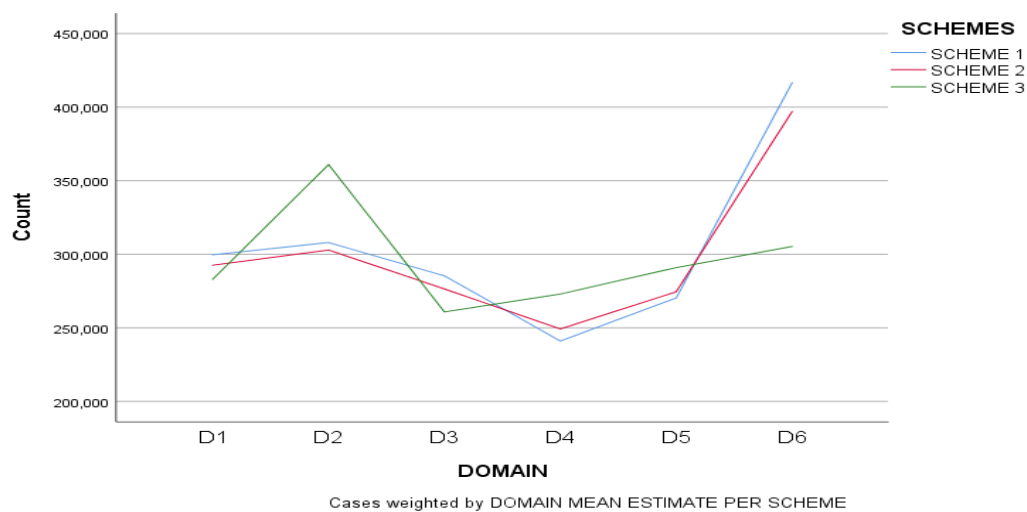
**Figure 1:** AMSE of the Estimators Per Scheme using Case 1 Across Domains and Quarters



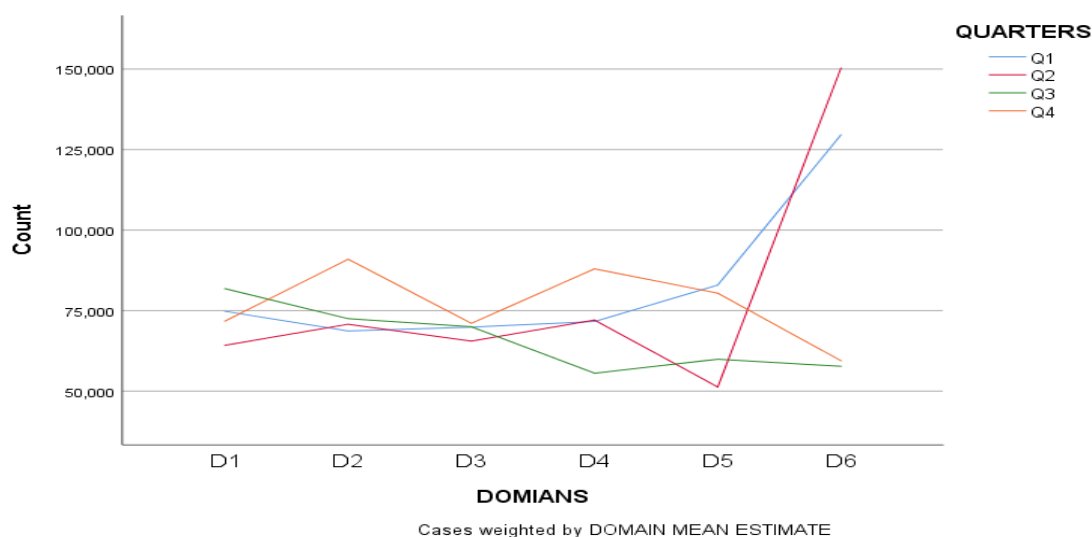
**Figure 2:** APRE of the Estimators Per Scheme using Case 1 Across Domains and Quarters



**Figure 3:** APRE of the Estimators Per Case Across the Domains and Quarters



**Figure 4:** Average Estimated Domain Expenditure Across the Quarters Per Scheme Using Case 1



**Figure 5:** Average Estimated Domain Expenditure Per Quarter Using Case 1

The results further revealed that both existing and proposed estimators achieved higher efficiency in Scheme 2, followed by Scheme 3 and there was a significant loss of efficiency in Scheme 1 across all domains and quarters. Again, comparing the proposed estimators in terms of the three schemes (scheme 1 -  $\bar{X}_d$  known and  $\bar{x}_{dn}$  utilized, Scheme 2 -  $\bar{X}_d$  known and  $\bar{x}_{dr}$  utilized and scheme 3 -  $\bar{X}_d$  unknown and  $\bar{x}_{dn}$  and  $\bar{x}_{dr}$  utilized) under the two cases, it was noticed that the estimators had outstanding performance in terms of gaining efficiencies in Scheme 2, followed by Scheme 3 and Scheme 1. By extension, using the mean of the response group produces the most efficient estimates, followed by Scheme 3 ( $\bar{X}_d$  unknown and  $\bar{x}_{dn}$  and  $\bar{x}_{dr}$  are utilized). Scheme 1 ( $\bar{X}_d$  known and  $\bar{x}_{dn}$  utilized) produced the least efficient estimates, though still effective for the proposed logarithmic imputation estimators. This implies that, imputing the missing values using the mean of the response group in the auxiliary variable, along with the population mean (if known), provides more efficient estimates. When the population mean is unknown, the sample mean serves as the second-best alternative. The Average Mean Squared Errors (AMSE) per estimator also confirmed that the proposed estimators perform better than the existing estimators across domains, schemes, and cases.

In Tables 9-14, it is observed that the Percentage Relative Efficiency (PRE) of the proposed estimators across the three schemes are greater than that of the existing estimators ( $T_{Bi}$ ,  $T_{Pi}$  and  $T_{DSi}$ ) across all domains and quarters. This was seen to be consistent in both response rate scenarios. The PRE in Case I (item response rate of 60%) was higher than in Case II (item response rate of 40%). This aligns with Bhushan et al. (2023), who found that estimators with higher response rates perform better than those with lower response rates. Hence, the proposed estimators outperform the existing estimators ( $T_{Bi}$ ,  $T_{Pi}$  and  $T_{DSi}$ ). This suggests that when the item response rate is high and consistent across subgroups, the estimator gains more efficiency. Similarly, the PRE of the proposed estimators under Case I was higher than under Case II, confirming that item nonresponse in survey sampling reduces PRE. Average Percentage Relative Efficiency (APRE) for all estimators showed that the proposed estimators perform better than the existing estimators across the domains for the four quarters.

The estimation of population means for Cases I and II as shown in Tables 15-16 revealed that the response rate has a greater influence on the estimation of the study variable. The parameter estimates (statistic) as well as the Average

Estimate (AE) for Case I in the four quarters were smaller than those of Case II. This implies that Case I estimates are more reliable in all six domains, four quarters, and schemes, since their MSE and AMSE were smaller than those of Case II, while the PRE and APRE were greater. The present findings differ from those reported by Iseh et al. (2025), who considered calibration estimation under double sampling with subsampling the nonrespondents, where subjecting either the study variable or both the study and auxiliary variables to nonresponse did not affect the population mean estimates. The difference may reflect the distinct sampling and estimation frameworks adopted in the two studies. This implies that response rate has a significant effect on estimating population means for the domain of study when imputation techniques are adopted to address item nonresponse, measurement error or missing information in single-stage sampling. Furthermore, the threat from nonresponse or missing information influences the MSE of the existing and proposed estimators.

The estimated household expenses using the single-stage design with respect to Scheme 2, Case 1 for the four quarters; Q<sub>1</sub>, Q<sub>2</sub>, Q<sub>3</sub>, and Q<sub>4</sub> are ₦82,908, ₦79,027, ₦66,246, and ₦71,925 respectively. This implies that the amount spent by households across the domains was highest in Quarter 1, followed by Quarter 2, while Quarter 3 recorded the lowest expenditure within the year. Similarly, the estimated expenses using the single-stage design with respect to Scheme 2, Case 1 for the six domains; D<sub>1</sub>, D<sub>2</sub>, D<sub>3</sub>, D<sub>4</sub>, D<sub>5</sub>, and D<sub>6</sub> are ₦73,149, ₦75,726, ₦69,108, ₦64,307, ₦62,307, and ₦99,252 respectively. This indicates that spending was highest in Domain 6 (South West), followed by Domain 2 (North East), while Domain 5 (South South) recorded the lowest expenditure within the year.

The results underscore the potential of logarithmic imputation in mitigating nonresponse bias. Unlike calibration and

reweighting, which rely on assumptions about response mechanisms, logarithmic imputation directly compensates for missing data while preserving distributional properties. The approach is particularly effective in domain estimation, where small sample sizes amplify the impact of nonresponse. Practical implications span labour force surveys, health studies, business surveys, and environmental monitoring contexts where domain-level accuracy informs policy and decision-making. By enhancing estimator precision, logarithmic imputation strengthens the reliability of survey-based evidence. The relatively high expenditure observed in Quarter 2, Domain 1, coincides with the reported income pattern; however, the present analysis does not establish a causal relationship between income and expenditure.

As seen in Figures 1-3, the proposed estimator outperformed existing estimators in terms of AMSE as well as APRE using the different schemes and the two cases, offering a solution to the persistent challenge of item nonresponse and extreme values in domains of study. The results indicate that the proposed estimators can provide more efficient domain mean estimates than the competing estimators considered under the examined under SRS and item-nonresponse scenarios. Again, in Figure 4, it is observed that Domain 6 (South West) had the highest household expenditure pattern, while Domain 4 (South East) recorded the lowest. Also in Figure 5, Quarters 1 and 2 recorded the overall highest household expenditure pattern, while quarters 3 and 4 recorded the overall lowest expenditure. This could be as a result of COVID-19 outbreak and Lockdown in 2020.

## 6.0 Conclusion

This study developed logarithmic imputation estimators for domain mean estimation with alternative auxiliary-information availability conditions within a single-stage simple random sampling framework in the presence of item nonresponse. First-order bias and mean squared

error expressions were derived for three imputation schemes, and their efficiency was compared with selected existing estimators. The empirical results indicate that the proposed estimators generally produced lower MSE and higher percentage relative efficiency under both the 60% and 40% item-nonresponse scenarios. Among the schemes examined, scheme 2 provided the strongest overall performance, followed by scheme 3, while scheme 1 generally recorded the lowest efficiency. The findings demonstrate that the use of logarithmic imputation with appropriate auxiliary information can improve the precision of domain mean estimates affected by item-nonresponse and outliers. However, the empirical conclusions are limited to the single-stage SRS framework and the response-rate scenarios examined in this study. Future research should investigate the proposed estimators under stratified and double-sampling designs, alternative nonresponse mechanisms, and broader empirical datasets.

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