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On the Additional Existence Conditions for t-Steiner Quintuple Systems of Balanced Incomplete Block Designs

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Abstract

A t -S (n, p, λ) balanced incomplete block design with the order 11 and cardinality 5 kept constant and t and λ rotated. Using combinatorics, 20 design tuples were generated. The Divisibility condition was then used in testing the design tuples and those that satisfied it were classified as Divisible Designs (DD) while those that did not satisfy it were classified as non-divisible designs (NDD). Explorations on both groups were carried out. The results show that designs exist from both the divisible group and the non-divisible group. These results indicate that the Divisibility Theorem cannot completely characterize the existence of the investigated t-Steiner Quintuple Systems, rather, it requires an additional existence condition. The possible additional existence conditions are proposed.

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1.0 Background

Steiner Systems is a system of balanced incomplete block designs which has received tremendous attention in the field of mathematics and statistics since it was first proposed and constructed by Sir Jacob Steiner in 1853. It can be said that Seiner Systems is the most studied structure in discrete mathematics due to its simplicity. Its history dates back to the work of Plucker (1835), Kirkman (1857), Cayley (1850), Bay and de Weck (1935), Bose (1939), Skolen (1958), Hannani (1961, 1975) and reported by (Encyclopedia of design theory, 2012). A block design is a non-empty N = $\{n_1, n_2, n_3 .... n_m\}$ whose elements are varieties and a non-empty collection of subsets of b blocks. When the copious numbers of similar experimental units are not accessible to contain all the treatments in block, then the use of incomplete block design is applied. Bose (1939). For note on the construction of balanced designs, see (Akra et al., 2021, 2024, 2025, 2025). A balanced incomplete block design (BIBD) is traditionally known by the parameters n, p, r, b, λ where n is the varieties or elements, p is the block size, r is the number each element is replicated in the blocks, b is the total number of blocks or block size and λ is the pair of treatment. These parameters are connected by the formula $bp = nr$ and $\lambda(n-1) = r(k-1)$ known as the Fisher's formula (Fisher, 1940). See Umoren and Isaac (2010), Isaac et al. (2025a, b) and Anthony (2018) for the construction of balanced designs. Similarly, Akra et al. (2021; 2023, 2025a, 2025b, 2025c) discussed different approaches for construction of BIBDs. Further discussions on the structures and applications of BIBD can be found in Michael et al. (2017, 2025), Usen et al. (2021), Muffat et al. (2025), Isaac et al. (2017) and Isaac and George (2025), Additional contributions are available in the works of Jungnickel and Tonchev (2019), Jungnickel et al. (2019), Colbourn and Rosa (1999) 2006) and Moura (2022).

1.1 Existence Conditions for BIBD and Steiner Systems

The main things about the Steiner systems and every other system of Balanced Incomplete Block Designs (BDID) are whether such a system exist, what are the conditions for its existence or existence conditions and finally, how is its constructed. Several authors have worked on and proposed the existence conditions and or additional existence conditions of the Steiner Systems and other systems of BIBD while others worked on the proof of the existence conditions. It could be noted that some of the existence conditions are necessary but not sufficient, thus, it requires additional existence condition or conditions.

1.1.1 Existence Conditions for Balanced Incomplete Block Design

The first to propose the existence condition for the BIBD which extends to the classical Steiner Systems is Fisher (1929) and studied by Yate (1936). He showed that for a BIBD, the parameters n, b, p , r and λ are connected by the formula $bp = nr$ and $\lambda(n-1) = r(p-1)$. Later, he proposed an additional existing condition for the parameters as $b \geq n$ (and hence $r \geq p$), $\lambda \geq 1$ and $k < v$ known as Fisher's inequality (Fisher, 1940). Balance Incomplete Block Designs with repeated blocks were studied and constructed initially by Van Lint & Ryser (1972) and pursued by Van Lint (1973). They showed that if m = b/n, then m divides gcb (b, r, λ). They also gave constructions for BIBDs with repeated blocks, usually with gcb (b, r, λ) > 1. They considered tuples (n, b, r, p, λ) of positive integers satisfying $2 \leq p \leq n/2$, $\lambda (n - 1) = r (p - 1)$, $nr = bp$, $\lambda > 1$, BIB(b, r, λ) = 1 and $b > 2n$, and asked whether for each such tuple (n, b, r, p, λ), there exists a BIBD with repeated blocks. He showed that this is indeed the case when $p \leq 4$, except possibly when (v, p, λ) = (45, 4, 3). Additionally, he tabulated all tuples (n, b, r, p, λ) satisfying these conditions with $n \leq 22$, and for many of these tuples constructed BIBDs with blocks. Seiden (1963) proved that there is no balanced incomplete block design with repeated blocks with parameters: n= 2k+2, b= 4k+2 and k= k+1 thus settling the case for general k not only for odd k. Stanton & Sprott (1964) showed that if S blocks of a balanced incomplete block design are identical, then $bn \geq sn - (s-2)$. Mann (1969) sharpened this result and showed that $b \geq sn$. Ho & Mendelsohn (1974) proposed the generalization of the Mann's inequality for t-designs. Van Lint & Ryser (1974) first gave a new proof of Mann's Inequality, stating that e < b/n if a block is repeated, e times, moreover, their methods allow them to conclude that equality is only possible if e/gcd (b, r, λ). They further showed that the number 'p' of distinct blocks satisfied p > n, with equality only if each block is repeated the same number of times, and also that $t \neq n+1$. Additionally, they constructed many examples of block designs, including several infinite families. Foody & Hedayat (1977) presented some potential applications of the balanced incomplete block designs to experimental designs and controlled sampling. They also provided some necessary and sufficient conditions for the existence of these designs and some algorithms for their constructions. He gave the necessary and sufficient condition under which a set of blocks can support a BIBD and a table of BIB designs with $22 \leq b^* \leq 56$ for n=8 and p=3 inclusive.

1.1.2 Existence Conditions for Steiner Systems

Kirkman (1847) is the one who proposed the first ever existing condition for the existence of the Steiner triple systems of order n. He said that the Steiner triple systems exist and is balanced if and only if n ≡ 1, 3 (mod 6). Steiner (1939) studied Kirkman's result independently and proved the necessary existence condition of S(n = 3) family using the same arguments as for BIBDs. He also showed that the total number of pairs, the total number of triples or blocks, b and the number of replication, r is $\frac{n(n-1)}{2}$, $\frac{n(n-1)}{6}$ and $\frac{n-1}{2}$ respectively. He equally showed that n must be of the form, 6k+*1* for some $k \geq 1$ if the design must exist. Bose (1939) contributed immensely to the study of S(n) by extending the construction of S(n) for n= 6k +3 for any natural number, k wherein he is known to have constructed S(9,3,1) design. Bose is reputed to be the first to propose the use of cyclic permutation to constructing S(n). Skolen (1959) on his part, improved upon the work of Bose and introduced the construction of n triple of type 1, 3n triple of type 2 and $\frac{3(2n)(2n-1)}{2} = 6n^2 - 3n$ triple of type 3. Moore (1896) that posed the problem of the existence of Steiner quadruple systems, SQS(v, k, λ) in which k = 4 in his paper, "Tactical Memoranda". Barrau (1908) worked on it and established the uniqueness and existence of the k = 4 family of BIBDs. He equally constructed S(8, 4, 3) and S(10, 4, 3) which is a 6n + 2 and 6n + 4 for n = 1 respectively. (Hanani, 1960) gave the necessary and sufficient conditions for the existence of such a system to be n ≡ 2 or 4 (mod 6) which can be interpreted as n = 6k + 2 or 6k + 4. This means that, unlike the triple system, n for the p= 4 systems must not be an odd number.

For the p =5 Steiner systems family known as Steiner Quintuple Systems, it was Hanani (1972) who gave the necessary but not sufficient condition for the existence of such a system as $n \equiv 5$ (mod 6) which comes from considerations that apply to all the classical Steiner systems. An additional necessary condition is that $n \neq 4$ (mod 5), which comes from the fact that the number of blocks must be an integer. Hannai (1975) also proved that for balanced incomplete block designs with blocks having five elements each (p = 5), the known necessary existence condition is also sufficient, with the exception of the non-existing design S(15, 5, 2). Nevertheless, the sufficient condition for the quintuple family of Steiner systems is still under investigation. This is so because there is a quintuple of order 11, that is, n = 6k + 5 for k = 1 but there is no quintuple of order 17, that is, n = 6k + 5 for k = 2. There are quintuple for order 23, that is for k = 3, but no quintuple for order 29, that is, k = 4, there is a quintuple for order 35, that is, for k = 5, but there is no quintuple for order 41, that is, k = 6 etc.

1.1.3 Existence conditions for the t- Steiner Systems Designs

While the parameters of all BIBDs including the Steiner systems BIBDs are supposed to incorporate t so that BIBD would be written as t-(n, p, λ) -BIB designs, yet, that was never the case. In those early designs, it was (n, p, b, r, λ) -BIBDs. It should be noted that while the relationship between n, b, p , r and λ had since been established by Fisher (1940) , the relationship between n, b, p, t, r and λ could not be found. Authors like Earl Krasmer & Dale Menser (1960), Wilson (1973) and Hanani (1975) attempted to fill this gap by linking the parameters of the BIBD with t and mentioned it passively but did not do an in-depth study on it.

The most useful necessary existence condition for the t- design was given by Keevash (2014) when he proposed the natural divisibility conditions for a t- (n, p, λ)-BIBD. He gave the divisibility condition for the existence of a t- Steiner Systems of balanced incomplete block design (Not t- Steiner Quintuple Systems only) as $\binom{p-i}{t-i}/\lambda\binom{n-i}{t-i}$. He further posited that the designs (tuples) that do not obey the divisibility theorem does not exist. In order to ascertain the veracity of the Divisibility Theorem especially if the design that do not obey it does not exist, Isaac et al (2026) and Isaac (2026) constructed a t-(11, 5, λ) Steiner systems for both the Divisible and Non-Divisible tuples of designs respectively. The results showed that there are divisible – existing and divisible- non –existing designs as well as non-divisible existing and non-divisible – non existing designs thus negating the belief that only Divisible designs exist. Furthermore, the discovery showed that the divisibility theorem, though, very efficient, yet, needs additional conditions to be imposed for greater efficiency. This is why this paper has become very necessary. This is what has motivated the curiosity to propose additional existing conditions to the Divisibility Theorem for t-designs. This paper is therefore aimed at proposing additional existing conditions to the Divisibility Theorem or condition for t-designs.

2.0 Materials and Methods

2. 1 t- Designs: A t - Design: A t-design or t – Steiner systems of balanced incomplete block design written as $t-(n,p,\lambda)$-BIBD design, is a pair (X,B), where X is a n-set of elements and B is a system of p-sets (called blocks) from X such that each t -set is in exactly λ.

2. 2 Necessary Condition for the Existence of t- Design

Theorem 1

Suppose that a t-(n, p, λ) design B exists with p = (p_1, . . ., p_m) and order n = (n_1, . . ., n_m), with |B| = b. Then, for any m-tuple t = (t_1, . . ., t_m) summing to t with $0 \le t_i \le k_i$ for i = 1, . . ., m, we have,

$$b\prod_{i=1}^{m}\binom{p-i}{t-i}=\lambda\prod_{i=1}^{m}\binom{n-i}{t-i}$$

2. 3 Divisibility Condition for the Existence of t- Design

Theorem 2: In a t-(n, p, λ)-BIBD,

$\binom{p-i}{t-i}$ is a divisor of $\lambda\binom{n-i}{t-i}$ for every

$$0 \le i \le t-1 \qquad (1)$$

Equation (1) is referred to as The Divisibility Theorem proposed by Peter Keevash (2014) for t-designs.

3.0 Methodology

3. 1 (a) Generation of Design Tuples

The first thing to generate is the design tuples and it is generated according to which parameter or parameters is or are varied and the one or ones that is or are kept constant. The parameters for a t – Steiner Quintuple system of balanced incomplete block design are (n, p, λ, t) written as **t- (n, p, λ)**.

In this work, t and λ will be rotated or varied while n and p will be kept constant, thus, for $2 \le t \le p$ and $\lambda = 1.....p$, the tuples are generated from the relationship **t-(n, p , λ)** thus;

$$t-\begin{pmatrix} n, & p & \lambda_1 \\ & & \lambda_2 \\ & & . \\ & & . \\ . & & \lambda_k \end{pmatrix} = t-\begin{pmatrix} n, & p & \lambda_1 \\ & & \lambda_2 \\ & & . \\ & & . \\ . & & \lambda_k \end{pmatrix},$$

$$t+1-\begin{pmatrix} n, & p & \lambda_1 \\ & & \lambda_2 \\ & & . \\ & & . \\ . & & \lambda_k \end{pmatrix} \quad ..... \quad t=p-\begin{pmatrix} n, & p & \lambda_1 \\ & & \lambda_2 \\ & & . \\ & & . \\ . & & \lambda_k \end{pmatrix}$$

This can also be given as:

$$= \; p-3-\begin{pmatrix} n, & p & \lambda_1 \\ & & \lambda_2 \\ & & . \\ & & . \\ . & & \lambda_k \end{pmatrix}, \; p-2-\begin{pmatrix} n, & p & \lambda_1 \\ & & \lambda_2 \\ & & . \\ & & . \\ . & & \lambda_k \end{pmatrix}$$

$$\ldots\ldots \; p-0-\begin{pmatrix} n, & p & \lambda_1 \\ & & \lambda_2 \\ & & . \\ & & . \\ . & & \lambda_k \end{pmatrix};$$

if p =i

3. 1 (b) Generation of blocks

To generate the blocks for design tuples, n combination p is used.

This is written mathematically as $\begin{pmatrix} n \\ p \end{pmatrix}$ or

$$\frac{(n)!}{(p)!(n-k)!} \qquad (2)$$

and the blocks will be given as:

$$\begin{matrix} b_1 & b_2 & b_3 & & b_{\binom{n}{p}} \\ \begin{pmatrix} 1 \\ 2 \\ . \\ . \\ p \end{pmatrix} & \begin{pmatrix} 1 \\ 2 \\ . \\ . \\ p \end{pmatrix} & \begin{pmatrix} 1 \\ 2 \\ . \\ . \\ p \end{pmatrix} & \ldots & .\begin{pmatrix} 1 \\ 2 \\ . \\ . \\ p \end{pmatrix} \end{matrix}$$

A design tuple is made up of several numbers of blocks of size, p and number of replications, r, with the following relationships which is generally referred to as the necessary conditions for BIBDs;

(i) $bp = nr$ (3)

(ii) $\lambda(n-1) = r(p-1)$ (4)

And for the t-Designs**,** the parameters are related thus;

$$b\begin{pmatrix} p \\ t \end{pmatrix} = \lambda\begin{pmatrix} n \\ t \end{pmatrix} \qquad (5)$$

where $b \geq n$ (and hence $r \geq p$) - (Fisher Inequality), $\lambda \geq 1$ and $p < n$

3. 2 Prove of Obedience / Non –obedience of Designs

After design tuples have been generated, each tuple has to be subjected to a test to prove whether it obeys the divisibility theorem or not using the equation (1). Those that obey or satisfy the divisibility theorem for t-designs are classified as obedient designs while those that do not obey the theorem are classified as non-obedient designs.

3.3 Construction of Designs

The for the recursive steps or algorithms for constructing a t – Steiner quintuple balanced incomplete block design, see Isaac et al (2026).

3.4 Construction of t-Steiner Quintuple Systems of Balanced Incomplete Block Designs

We propose to construct some t- Steiner quintuple of balanced incomplete block designs of order 11 written as t- SQS(11, 5, λ). In this study, **t and** λ shall be varied while v and k shall be kept constant.

The design tuples for t –(11, 5 , λ) are;

$2-(11,5,1)$	$3-(11,5,1)$
$2-(11,5,2)$	$3-(11,5,2)$
$2-(11,5,3)$	$3-(11,5,3)$
$2-(11,5,4)$	$3-(11,5,4)$
$2-(11,5,5)$	$3-(11,5,5)$

$4-(11,5,1)$	$5-(11,5,1)$
$4-(11,5,2)$	$5-(11,5,2)$
$4-(11,5,3)$	$5-(11,5,3)$
$4-(11,5,4)$	$5-(11,5,4)$
$4-(11,5,5)$	$5-(11,5,5)$

Testing for divisibility and non-divisibility was carried out using equation (1) and the results tabulated in Table 1 and Table 2 respectively.

Table 1: Divisible Designs

S/N	No of Pairs (t)	t-designs Type
1	2	2-(11, 5, 2)
2		2-(11, 5, 4)
3	3	3-(11, 5 , 2)
4		3-(11, 5, 4)
5	4	4-(11, 5, 1)
6		4-(11, 5, 2)
7		4-(11, 5, 4)
8		4-(11, 5, 5)
9	5	5- (11, 5, 1)
10		5- (11, 5, 2)
11		5- (11, 5, 3)
12		5- (11, 5, 4)
13		5- (11, 5, 5)

Table 2: Non- Divisible Designs

S/N	No of Pairs (t)	t-designs Type
1	2	2-(11, 5, 1)
2		2-(11, 5, 3)
3		2-(11, 5, 5)
4	3	3-(11, 5, 1)
5		3-(11, 5, 3)
6		3-(11, 5, 5)
7	4	4-(11, 5, 3)

1. Construction and Analysis of the Divisible Group of Designs

The divisibility theorem posited that all the Divisible Designs tuples exist and can be constructed However, it was discovered that of the thirteen (13) divisible designs, only six (6) exist with their number of blocks and number of replications given while seven (7) do not exist. The result is in table 3 and table 4.

(2) Construction and Analysis of the Non-divisible Group

The divisibility theorem posited the all-non-divisible designs do not exist; however, it was discovered that four (4) out of the seven designs exist and were constructed (Isaac, 2026). The results are given in table 5 and table 6.

In order to make some additional theorems to the Divisibility Theorem on the existing of t- S (11, 5, λ)- BIBDs, all the existing and non-existing designs are hereby tabulated in table 7 and table 8.

Table 3: Divisible Existing Designs with their Number of Blocks and Number of Replication

S/N	No of Pairs	t-designs Type	No of Blocks	No of Groups of 11	No of replications
1	2	2-(11, 5, 2)	11	1	5
2		2-(11, 5, 4)	22	2	10
3	3	3-(11, 5 , 2)	33	2	15
3	4	4-(11, 5, 1)	66	6	30
5		4-(11, 5, 2)	132	12	60
6	5	5- (11, 5, 1)	11	1	5

Table 4: Divisible non-existing designs

S/N	No of Pairs (t)	t-designs Type
1	3	3-(11, 5 , 4)
2	4	4-(11, 5, 4)
3		4-(11, 5, 5)
4	5	5- (11, 5, 2)
5		5- (11, 5, 3)
6		5- (11, 5, 4)
7		5- (11, 5, 5)

Table 5: Non- Divisible Existing Designs

S/N	No of Pairs (t)	t-designs Type
1	2	2-(11, 5, 1)
2		2-(11, 5, 3)
3	3	3-(11, 5, 1)
4		3-(11, 5, 3)

Table 6: Non-Divisible –Non -Existing Designs

S/N	No of Pairs (t)	t-designs Type
1	2	2-(11, 5, 5)
2	2	3-(11, 5, 5)
3	3	4-(11, 5, 3)

Table 7: All Existing t- S (11, 5, λ) Designs

S/N	Divisible Designs	Non-Divisible Designs
1	2-(11, 5, 2)	2-(11, 5, 1)
2	2-(11, 5, 4)	2-(11, 5, 3)
3	3-(11, 5 , 2)	3-(11, 5, 1)
4	4-(11, 5, 1)	3-(11, 5, 3)
5	4-(11, 5, 2)	
6	5- (11, 5, 1)	

Table 8: All Non –Existing t- S(11, 5, λ) Designs

S/N	Divisible Designs	Non-Divisible Designs
1	3-(11, 5 , 4)	2-(11, 5, 5)
2	4-(11, 5, 4)	3-(11, 5, 5)
3	4-(11, 5, 5)	4-(11, 5, 3)
4	5- (11, 5, 2)	
5	5- (11, 5, 3)	
6	5- (11, 5, 4)	
7	5- (11, 5, 5)	

Proposed Theorems.

Theorem 1: In a t-(11, 5, λ), $t \le p$ .designs exist.

Proof

From Equation (5), we have that

$$b\binom{p}{t} = \lambda\binom{n}{t}.$$

But $$\binom{p}{t} = \frac{p!}{t!(p-t)!} \tag{6}$$

But (p-t) must not be < 1, otherwise, equation (6) will become undefined. This can only happen if $t \le p$. Thus, in a t-(11, 5, λ), $t \le p$.

Theorem 2: In a t-(n, p, λ), b or block size = $\lambda\binom{n-i}{t-i}/\binom{p-i}{t-i}$ for i = 0 and t = 2

Proof

From Equation (5), we have

$$b\binom{p-i}{t-i} = \lambda\binom{n-i}{t-i}.$$

This implies that

$$b = \lambda\binom{n-i}{t-i}/\binom{p-i}{t-i}$$

And if i = o, then we have

b = $\lambda\binom{n}{t}/\binom{p}{t}$

If we put t = 1, then we have

b = $\lambda\binom{n}{1}/\binom{p}{1}$ which reduces to

b = $\frac{n\lambda}{p}$

But combining equation 4 and 5, we have

b = $\frac{n\lambda}{p}.\frac{n-1}{p-1}$ (for all BIBDs)

$$\lambda\binom{n}{t}/\binom{p}{t} = \lambda\frac{n!}{t!(n-t)!}/\frac{p!}{t!(p-t)!}$$

$$= \lambda\frac{n(n-1)(n-2)!}{t!(n-t)}/\frac{p(p-1)(p-2)!}{t!(p-t)}$$

$$= \lambda\frac{n(n-1)(n-2)!}{p(p-1)(p-2)!}x\frac{t!(p-t)!}{t!(p-t)!}$$

As t = 2 (min value), the equation reduces to

b = $\lambda\frac{n(n-1)}{p(p-1)}$. Hence obeys the condition of a BIBD.

Theorem 3: In a t-(11, 5, λ), t + λ ≤ 6 designs exist and t + λ > 6 designs does not exist (from Tables 7 and 8).

4.0 Summary of findings and Conclusions

4. 1. Discussion of Findings

A t -S (n, p, λ) balanced incomplete block design with the order 11 and cardinality 5 kept constant and t and λ rotated. Using combinatorics, 20 design tuples were generated. Divisibility condition was then used in testing the design tuples and those that satisfied it were classified as Divisible Designs (DD) while those that did not satisfy it were classified as non-divisible designs (NDD). Isaac et al (2026) carried out construction on the divisible group and gave the results in table 3 and table 4 respectively. Isaac (2026) carried out exploration and construction on the non-divisible group and equally gave the results in table 5 and table 6 respectively. Table 7 and table 8 show all the existing and non-existing designs from both groups respectively. Looking at the results in table 7 and table 8 respectively, it can be posited that the divisibility theorem is efficient but requires additional existence conditions to improve its efficiency. The reason being that without imposing an additional existing condition, only seven (7) tuples would have been constructed while a whooping four (4) tuples would have been thrown away. Again, by imposing an additional condition, the time used in searching for a whopping ten (10) non existing designs would be saved. Mathematically and statistically, that will improve the efficiency of the divisibility theorem. This is why the proposed additional existing condition to the Divisibility Theorem has become very necessary.

4. 2 Conclusion

From the findings of this research above, we can propose two existence conditions for a t-S(n, p, λ) designs in addition to the divisibility condition namely;

1. A t- S (11, 5, λ) exists for $2 \le t \le 5$ and $1 \le \lambda \le 4$. This means that a design with t up to p exists but designs with λ up to p does not exist.
2. A design with t +λ ≤ 6 exists while designs with t + λ > 6 does not exist.

To this end, the divisibility condition or theorem can be given as;

$\binom{p-i}{t-i}$ is a divisor of $\lambda\binom{n-i}{t-i}$ for every

$$0 \le i \le t-1$$

with the following additional conditions;

i. t- S (11, 5, λ) exists for $2 \le t \le 5$ and $1 \le \lambda \le 4$.
ii. A design with t + λ $\le$ 6 exists.
iii. A design with t + λ > 6 does not exist.

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Conflict of Interest
The authors assert that there is no conflict of interest whatsoever.

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