



On the Existence and Construction of Some Non-Divisible t -Steiner Quintuple Systems of Balanced Incomplete Block Designs

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Abstract

This study investigates the existence and construction of some non-divisible t -Steiner quintuple systems of balanced incomplete block designs (BIBDs). Design tuples of the parameters t - (n, p, λ) for t and λ varied, and n , the order, and p , the cardinality, kept constant at eleven (11) and five (5) respectively were generated and tested using the divisibility theorem. The design tuples that satisfied the divisibility theorem were classified as Divisible Designs while those that did not satisfy the divisibility theorem were classified as non-divisible designs. Interest was on the group of non-divisible designs and the interest was to ascertain if all non-divisible designs do not exist in line with the divisibility theorem. In order to obtain the result, some divisible designs were constructed and their incidence matrices shown. These incidence matrices were then observed to also serve as the incidence matrices of the non-divisible designs. The results showed that some non-divisible design tuples exist. The result further showed that there exist structural relationships between the Divisible and some Non-divisible designs.

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1.0 Introduction

A block design is a non-empty $N = \{n_1, n_2, n_3, \dots, n_m\}$ whose elements are varieties and a non-empty collection of subsets of b blocks. When the copious numbers of similar experimental units are not accessible to contain all the treatments in block, then the use of incomplete block design is applied Bose (1939). For note on the construction of balanced designs, see (Akra et al., 2024, 2025). A balanced incomplete block design (BIBD) is traditionally known by the parameters n, p, r, b, λ where n is the varieties or elements, p is the block size, r

is the number each element is replicated in the blocks, b is the total number of blocks or block size and λ is the pair of treatment. Studies by Umoren and Isaac (2010), Isaac et al. (2025a, b) and Anthony (2018) provide useful insights into the construction of balanced designs. Similarly, Akra et al. (2021, 2023, 2025) discussed different approaches for constructing Balanced Incomplete Block Designs (BIBDs). Further discussions on the structure and applications of BIBDs can be found in Michael et al. (2017, 2025), Usen et al. (2021), Muffat et al. (2025), Isaac et al. (2017), and Isaac and George (2025). Additional foundational contributions are also

available in the works of Jungnickel and Tonchev (2019), Jungnickel et al. (2019), Colbourn and Rosa (1999), Colbourn and Mathon (2006), as well as Moura (2022).

1.1 Steiner systems

Steiner systems of balanced incomplete block designs are special designs of order n , written as STS (n) , and is defined as a (N, B) - BIBD with N points of set S elements with a set of p elements which are subset of N called blocks and each block $b, \in B$ has $|B| = p$ such that each pair of distinct elements $x, y \in N$ occur together in exactly λ blocks. Steiner systems are called after the value of the block size or cardinality p . It was first proposed and constructed by Kirkman in 1857. Steiner (1939) studied the triple system, STS $(n, 3, \lambda)$ and proposed the problem of arranging 'n' objects in triplets such that every pair of objects appears in precisely one triplet. That arrangement is a balanced incomplete block design and because his work was more broadly disseminated in mathematical circles, the triple system was named after him. He proved the necessary existence condition of STS (n) using the same arguments as for BIBDs and also showed that the total number of pairs, the total number of triples or blocks, b and the number of replication, r is $\frac{n(n-1)}{2}$, $\frac{n(n-1)}{6}$ and $\frac{n-1}{2}$ respectively. He equally showed that n must be of the form, $6k+1$ for some $k \geq 1$ if the design must exist.

The work by Hannani (1960) opened the floodgates for several researches on the $p = 4$ Steiner systems. It also raised curiosity on the study of the quintuple or $p = 5$ family of the Steiner systems called the Steiner quintuple systems. A Steiner System, $S(n, p, \lambda)$ - BIBDs in which p or cardinality or block size is 5 is called a Steiner quintuple system and there are quite a few literatures in the quintuple systems of balanced incomplete design. Again, the man behind the introduction of the $p = 5$ Steiner systems is Hanani (1972) and because it follows

the Steiner systems, it was called the Steiner quintuple systems written as STS $(n, 5, \lambda)$. He gave the necessary (not sufficient) condition for the existence of such a system as $n \equiv 5 \pmod{6}$ which comes from considerations that applies to all the classical Steiner systems. An additional necessary condition is that $n \not\equiv 4 \pmod{5}$, which comes from the fact that the number of blocks must be an integer. He proved that for balanced incomplete block designs with blocks having five elements each ($p = 5$), the known necessary existence condition is also sufficient, with the exception of the non-existing design $S(15, 5, 2)$. Nevertheless, the sufficient condition for the quintuple family of Steiner systems is still under investigation. This is so because there is a quintuple of order 11, that is, $n = 6k + 5$ for $k = 1$ but there is no quintuple of order 17, that is, $n = 6k + 5$ for $k = 2$. There are quintuple for order 23, that is for $k = 3$, but no quintuple for order 29, that is, $k = 4$, there is a quintuple for order 35, that is, for $k = 5$, but there is no quintuple for order 41, that is, $k = 6$ etc.

1.2 t-Designs

It has been established that in a BIBD, every pair $\{x, y\}$ appears together λ times. This pair is what is referred to as t . Though initially, this pair was equal to two in the early designs like in Kirkman (7,3,1), Bose (9,3,1), Steiner (7, 3,1), Kirkman (7, 3, 2) and Denniston (15, 3, 2) designs, in Bose (1939, 1942), Kirkman (1857) and Denniston (1974) respectively. Yet, since it was not regarded as an important parameter, it was not attached to other BIBD parameters. While the parameters of all BIBDs including the Steiner systems BIBDs are supposed to incorporate t so that BIBD would be written as t - (n, p, λ) -BIB, yet, that was never the case. In those early designs, it was (n, p, b, r, λ) -BIBDs. Several works on t - designs have been studied by different authors like Hanani (1972), Wilson (1973), Ho and Mendelson (1974), and Earl and Dale (1976). However, a breakthrough came when Keevash (2014) proposed the natural

divisibility conditions for a t - (n, p, λ) -BIBD. He posited that a design that does not satisfy the divisibility condition does not exist. Isaac et al (2026) investigated the t –Steiner quintuple systems by varying t and λ while keeping n and p constant at eleven (11) and five (5) respectively. He showed that thirteen (13) designs tuples out of twenty (20) satisfied the divisibility condition for t -designs but that only six (6) out of the thirteen (13) designs exists, which he classified as Obedient (divisible) - Existing design tuples. This discovery that there are divisible – Existing and divisible non – Existing designs is what raises the curiosity to investigate if there are existing non - divisible designs and non – existing – non – divisible designs tuples.

This paper is therefore aimed at verifying if the seven (7) designs tuples out of the twenty (20) tuples generated from t - $(11, 5, \lambda)$ that satisfied not the divisibility condition for t - Steiner quintuple designs does not exist and hence cannot be constructed in line with the divisibility theorem or condition or whether some exist and can be constructed.

2.0 Materials and Method

2.1 The Necessary Condition for the Existence of t-Design

Theorem 1

Suppose that a t - (n, p, λ) design B exists with $p = (p_1, \dots, p_m)$ and order $n = (n_1, \dots, n_m)$, with $|B| = b$. Then, for any m -tuple $t = (t_1, \dots, t_m)$ summing to t with $0 \leq t_i \leq k_i$ for $i = 1, \dots, m$, we have,

$$b \prod_{i=1}^m \binom{p-i}{t-i} = \lambda \prod_{i=1}^m \binom{n-i}{t-i} \tag{1}$$

Theorem 2: In a t - (n, p, λ) - BIBD,

$$\binom{p-i}{t-i} \text{ is a divisor of } \lambda \binom{n-i}{t-i} \text{ for every } 0 \leq i \leq t-1 \tag{2}$$

Equation (2) is referred to as the Divisibility Theorem. The divisibility theorem posits that all

designs that satisfy or obey it exists and all the designs that do not satisfy or obey it does not exist.

2.2. Generation of Design Tuples

The first thing is to generate design tuples and they are generated according to which parameter or parameters is or are varied and the one or ones that is or are kept constant. In this work, t and λ will be rotated or varied while n and p will be kept constant, thus, for $2 \leq t \leq p$ and $\lambda = 1, \dots, p$ the tuples are generated from the relationship; t - (n, p, λ) thus;

$$t - \begin{pmatrix} n, & p & \lambda_1 \\ & & \lambda_2 \\ & & \cdot \\ & & \cdot \\ & & \lambda_k \end{pmatrix} = t - \begin{pmatrix} n, & p & \lambda_1 \\ & & \lambda_2 \\ & & \cdot \\ & & \cdot \\ & & \lambda_k \end{pmatrix} =$$

$$t+1 - \begin{pmatrix} n, & p & \lambda_1 \\ & & \lambda_2 \\ & & \cdot \\ & & \cdot \\ & & \lambda_k \end{pmatrix} \dots t = p - \begin{pmatrix} n, & p & \lambda_1 \\ & & \lambda_2 \\ & & \cdot \\ & & \cdot \\ & & \lambda_k \end{pmatrix}$$

This can also be given as:

$$p-3 - \begin{pmatrix} n, & p & \lambda_1 \\ & & \lambda_2 \\ & & \cdot \\ & & \cdot \\ & & \lambda_k \end{pmatrix}, \quad p-2 - \begin{pmatrix} n, & p & \lambda_1 \\ & & \lambda_2 \\ & & \cdot \\ & & \cdot \\ & & \lambda_k \end{pmatrix}$$

$$\dots \quad p-0 - \begin{pmatrix} n, & p & \lambda_1 \\ & & \lambda_2 \\ & & \cdot \\ & & \cdot \\ & & \lambda_k \end{pmatrix}; \text{ if } p = i$$

2.3 Test of Design Tuples on Divisibility Theorem

After design tuples have been generated, each tuple has to be first subjected to a test to ascertain if it obeys or satisfies the divisibility theorem or not using equation (2). Those that obey or satisfy the divisibility theorem for t -designs are classified as Divisible Designs (DD)

while those that do not obey the theorem are classified as Non- divisible Designs (NDD).

2.4 Generation of blocks

The blocks of the design’s tuples are generated by method of combinatorics thus we have n combination p. This is written mathematically as:

$$\binom{n}{p} \text{ or } \frac{(n)!}{(p)!(n-p)!} \tag{3}$$

and the blocks will be given as:

$$\begin{matrix}
 b_1 & b_2 & b_3 & \dots & b_{\binom{n}{p}} \\
 \begin{pmatrix} 1 \\ 2 \\ \cdot \\ \cdot \\ p \end{pmatrix} & \begin{pmatrix} 1 \\ 2 \\ \cdot \\ \cdot \\ p \end{pmatrix} & \begin{pmatrix} 1 \\ 2 \\ \cdot \\ \cdot \\ p \end{pmatrix} & \dots & \begin{pmatrix} 1 \\ 2 \\ \cdot \\ \cdot \\ p \end{pmatrix}
 \end{matrix}$$

A design tuple is made up of several numbers of blocks, b of size, p and the number of replications, r, n and λ with the following relationships which are generally referred to as Fisher’s necessary conditions for the existence of BIBDs Fisher (1940):

$$(i) \quad bp = nr \tag{4}$$

$$(ii) \quad \lambda(n-1) = r(p-1) \tag{5}$$

And for the t-Designs, the parameters are related thus;

$$b \binom{p}{t} = \lambda \binom{n}{t} \tag{6}$$

where $b \geq n$ (and hence $r \geq p$) - (Fisher Inequality) $\lambda \geq 1$ and $p < n$.

2.5 Construction of Designs

To construct a design is the last step in design’s construction. This has to do with arranging together the blocks that suit or gives the design. Sometimes a design may exist and sometimes, it may not exist. After arranging these blocks, the design is tested using the incidence matrix (see Isaac et al. (2026) for the recursive steps in constructing designs).

3.0 Results

The following are the results obtained when a t – (11, 5, λ) tuples of designs were generated by varying t and λ while keeping n and p constant. It is to be noted that the generation of tuples, test on divisibility theorem, generation of blocks cyclically, and construction of incidence matrix for each design are done manually using combinatorics.

3.1 Generation of Design tuples for t – (11, 5, λ)

The following designs tuples were generated. They are twenty (20) in number. They are;

- | | |
|-----------------|-----------------|
| 2 – (1 1, 5, 1) | 3 – (1 1, 5, 1) |
| 2 – (1 1, 5, 2) | 3 – (1 1, 5, 2) |
| 2 – (1 1, 5, 3) | 3 – (1 1, 5, 3) |
| 2 – (1 1, 5, 4) | 3 – (1 1, 5, 4) |
| 2 – (1 1, 5, 5) | 3 – (1 1, 5, 5) |
| 4 – (1 1, 5, 1) | 5 – (1 1, 5, 1) |
| 4 – (1 1, 5, 2) | 5 – (1 1, 5, 2) |
| 4 – (1 1, 5, 3) | 5 – (1 1, 5, 3) |
| 4 – (1 1, 5, 4) | 5 – (1 1, 5, 4) |
| 4 – (1 1, 5, 5) | 5 – (1 1, 5, 5) |

3.2 Test of Designs on the Divisibility Theorem

Using equation (2), it here shown how the design tuples are tested on the divisibility theorem or condition.

$$\binom{p-i}{t-i} = \lambda \binom{n-i}{t-i} ; \text{ for every } 0 \leq i \leq t-1$$

Example 1

For the tuple, t=2, λ=2, n=11 and p=5 written as 2-(11, 5, 2)

$$(n) = 11 = \{0,1,2,3,4,5,6,7,8,9,10\}$$

Since, t= 2, i = 0, 1., p = 5, λ =2

for i = 0,

$$\binom{5}{2} = 2 \binom{11}{2}$$

$$10 = 2 \cdot 11 \cdot 5$$

⇒ the RHS is a divisor of the LHS.

for i = 1,

$$\binom{4}{1} = 2 \binom{10}{1}$$

$$4 = 2.10$$

\Rightarrow the RHS is a divisor of the LHS.

Since the two systems showed that the RHS is a divisor of the LHS, we conclude that a 2-(11, 5, 2) design satisfies the divisibility theorem.

Example 2

For the tuple, $t=2$, $\lambda=3$, $n=11$ and $p=5$ written as 2-(11, 5, 3)

$$(n) = 11 = \{0,1,2,3,4,5,6,7,8,9,10\}$$

Since, $t=2$, $i=0, 1$, $p=5$, $\lambda=3$

for $i=0$,

$$\binom{5}{2} = 3 \binom{11}{2}$$

$$10 = 3. 11. 5$$

\Rightarrow the RHS is not a divisor of the LHS.

for $i=1$,

$$\binom{4}{1} = 3 \binom{10}{1}$$

$$4 = 3.10$$

\Rightarrow the RHS is not a divisor of the LHS.

Since the two systems showed that the RHS is not a divisor of the LHS, we conclude that a 2-S (11, 5, 3) does not satisfy the divisibility theorem. The twenty tuples are tabulated into DD and NDD as shown in Tables 1 and 2.

Table 1: Design that satisfies the Divisibility Theorem (Divisible Design, DD)

S/N	No of Pairs (t)	t-designs Type
1	2	2-(11, 5, 2)
2		2-(11, 5, 4)
3	3	3-(11, 5, 2)
4		3-(11, 5, 4)
5	4	4-(11, 5, 1)
6		4-(11, 5, 2)
7		4-(11, 5, 4)
8		4-(11, 5, 5)
9	5	5- (11, 5, 1)
10		5- (11, 5, 2)
11		5- (11, 5, 3)
12		5- (11, 5, 4)
13		5- (11, 5, 5)

Table 2: Design that does not satisfy the Divisibility Theorem (Non –divisible Design, NDD)

S/N	No of Pairs (t)	t-designs Type
1	2	2-(11, 5, 1)
2		2-(11, 5, 3)
3		2-(11, 5, 5)
4	3	3-(11, 5, 1)
5		3-(11, 5, 3)
6		3-(11, 5, 5)
7	4	4-(11, 5, 3)

3. 3. Exploration on the Group of Non-divisible Designs

To construct a design, we first generate the blocks from equation (3). In this study, the total

number of blocks is given as $\binom{n}{p}$ or

$$\frac{(n)!}{(p)!(n-p)!} = \binom{11}{5} = 462 \text{ blocks, each of size}$$

5. For the algorithm on design construction, see Isaac et al. (2026).

1. A 3-(11, 5, 1) - BIBD which is a non-divisible design and which axioms for a BIBD does not hold as well, yet, from block (02348) as the base block for a 2-(11, 5, 2)- BIBD, generating other blocks cyclically and constructing its incidence matrix, we have;

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \end{bmatrix}$$

From the incidence matrix above, $P(t_3, t_2, t_1), P(t_4, t_3, t_2), \dots, P(t_{10}, t_9, t_8), \lambda = 1, p = 5, r = 5,$ and $n = 11$ is a 3 - (11,5,1)- BIBD. However, from the above incidence matrix, it could be seen hat the same incidence matrix of a 3 - (11, 5, 1) is also the incidence matrix for 2 - (11, 5, 2). Hence, there is a relationship between the two designs as follows (Table 3);

Table 3: Table showing Relationship when a 3 - (11,5, 1) Design is Constructed

S/N	t-design tuple	Type
1	3-(11, 5, 1)	Non-Divisible
2	2-(11, 5, 2)	Divisible

2. A 5-(11, 5, 1)- BIBD which is a Divisible design and which axioms for a BIBD hold

generated from block (018910) as the base block, generating other blocks cyclically and constructing its incidence matrix, we have;

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

From the incidence matrix above, it could be seen that a 5-(11,5,1) designs is a 4 - (11, 5, 2) design, a 3 - (11, 5, 3) design and a 2 - (11, 5, 2) design (Table 4).

Table 4: Table showing Relationship when a 5- (11,5,1) Design is Constructed

S/N	t-design tuple	Type
1	2-(11, 5, 4)	Divisible
2	3-(11, 5, 3)	Non-divisible
3	4-(11, 5, 2)	Divisible
4	5-(11, 5, 1)	Divisible

3. A 4-(11, 5, 1)- BIBD which is a Divisible design and which axioms for a BIBD hold generated from block (04567) as the base block, generating other blocks cyclically and constructing its incidence matrix, we have;

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

From the incidence matrix above, we see that a 4 - (11, 5, 1) is a 3 - (11, 5, 2) and a 2-(11,5,3) designs (Table 5).

4.0 Discussion and Conclusion

4.1 Discussion

Twenty (20) design tuples from t -STS $(11, 5, \lambda)$ were generated and tested using the divisibility theorem. The Twenty (20) design tuples were classified as Divisible design (DD) and Non-divisible designs (NDD). Of the Twenty (20) design tuples, thirteen (13) tuples were DD while seven (7) design tuples were NDD (Table 5).

Table 5: Table showing Relationship when a 4- $(11, 5, 1)$ Design is Constructed

S/N	t -design tuple	Type
1	2- $(11, 5, 3)$	Non-Divisible
2	3- $(11, 5, 2)$	• Divisible
3	4- $(11, 5, 1)$	• Divisible

Interest was on the group of non-divisible (non-obedient as per Isaac et al 2026) designs and the interest was to ascertain if all non-divisible designs do not exist according to the divisibility theorem. In order to obtain the result, some divisible designs were constructed and their incidence matrices shown. These incidence matrices were observed to also serve as the incidence matrices of the non-divisible designs. For example, in constructing a 3- $(11, 5, 2)$ divisible design, its incidence matrix showed that the same is 2- $(11, 5, 3)$ non-divisible design. In other words, a construction of a 3- $(11, 5, 2)$ divisible design, is the construction of 2- $(11, 5, 3)$ non-divisible design. In the same vein, a 5- $(11, 5, 1)$ divisible design is a 3- $(11, 5, 3)$ non-divisible design. After all the searches were carried out on all the divisible designs, result shows that three (3) out of the seven (7) Non-divisible designs namely 2- $(11, 5, 3)$, 3- $(11, 5, 1)$, and 3- $(11, 5, 3)$ exists and constructible while four (4) designs namely 2- $(11, 5, 1)$, 2- $(11, 5, 5)$, 3- $(11, 5, 5)$, and 4- $(11, 5, 3)$ do not exist and hence cannot be constructed. Furthermore, results show that there exist interesting structural relationships between divisible and non-divisible designs. For instance, a 5- $(11, 5, 1)$

design is also a 4- $(11, 5, 2)$ design both of which are divisible designs and is also a 3- $(11, 5, 3)$ design which is a non-divisible design. These relationships are the backbone for the existence and construction of some of these non-divisible designs which hitherto were considered to be non-existing designs. A backward and forward relationships for t and λ in a t -Steiner quintuple systems exist. A t - $S(n, 5, \lambda)$ is a $t+1$ - $S(n, 5, \lambda-1)$, a $t+2$ - $S(n, 5, \lambda-2)$ and a $t+3$ - $S(n, 5, \lambda-3)$. For the backward relationship, a t - $S(n, 5, \lambda)$ is a $t-1$ - $S(n, 5, \lambda+1)$, a $t-2$ - $S(n, 5, \lambda+2)$ and a $t-3$ - $S(n, 5, \lambda+3)$. A 5- $(11, 5, 1) = 4$ - $(11, 5, 2) = 3$ - $(11, 5, 3) = 2$ - $(11, 5, 4)$. A 4- $(11, 5, 1) = 3$ - $(11, 5, 2) = 2$ - $(11, 5, 3)$. Similarly, a 3- $(11, 5, 1) = 2$ - $(11, 5, 2)$ designs.

4.2 Conclusion

From these findings, it could be concluded that some Non-divisible designs exist and can be constructed against the earlier notion that all designs, as long as they satisfy not the divisibility theorem, are non-existing designs. This result is made possible because of the discovery that there are some relationships between Divisible and Non-divisible designs. It has been established that not all Divisible designs exist. Divisible designs that do not exist are called Divisible-Non-Existing designs (Isaac et al., 2026). In the same vein, findings of this research have shown that not all non-divisible designs exist, at least four (4) do not exist. Findings further show that what gives rise to the existence and construction of some of the non-divisible designs is the existence of these structural relationships. For instance, the construction of a 5- $(11, 5, 1)$ design is also the construction of a 4- $(11, 5, 2)$, a 3- $(11, 5, 3)$ and a 2- $(11, 5, 4)$ designs. In the same vein, the construction of a 4- $(11, 5, 1)$ design is also the construction of a 3- $(11, 5, 2)$ and a 2- $(11, 5, 3)$ designs.

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Conflict of Interest

The author asserts that there is no conflict of interest whatsoever.

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